

# CATEGORICITY AND SOLVABILITY OF A.E.C., QUITE HIGHLY

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## §0 INTRODUCTION

The hope which motivates this work is

0.1 Conjecture: If  $\mathfrak{K}$  is an a.e.c. then either for every large enough cardinal  $\mu$ ,  $\mathfrak{K}$  is categorical in  $\mu$  or for every large enough cardinal  $\mu$ ,  $\mathfrak{K}$  is not categorical in  $\mu$ .  
Why do we consider this a good dream? See Chapter N.

Our main result is 4.10, it says that if  $\mathfrak{K}$  is categorical in  $\mu$  (ignoring few exceptional  $\mu$ 's) and  $\lambda \in [\text{LS}(\mathfrak{K}), \mu)$  has countable cofinality and is a fix point of the sequence of the  $\beth_\alpha$ 's, (moreover a limit of such cardinals) then there is a superlimit  $M \in K_\lambda$  for which  $\mathfrak{K}_{[M]} = \mathfrak{K}_\lambda \restriction \{M' : M' \cong M\}$  has the amalgamation property (and a good  $\lambda$ -frame  $\mathfrak{s}$  with  $\mathfrak{K}_\mathfrak{s} = \mathfrak{K}_{[M]}$ ). Note that Chapter IV seems to give a strong indication that finding good  $\lambda$ -frames is a significant advance. This may be considered an unsatisfactory evidence of an advance, being too much phrased in the work's own terms. So we prove in §5 - §7 that for a restrictive context we make

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a clear cut advance: assuming amalgamation and enough instances of  $2^\lambda < 2^{\lambda^+}$  occurs, much more than the conjecture holds, see Chapter N on background.

Note that as we try to get results on  $\lambda = \beth_\lambda > \text{LS}(\mathfrak{K})$ , clearly it does not particularly matter if for  $\kappa \in (\text{LS}(\mathfrak{K}), \lambda)$  we use, e.g.  $\kappa_1 = \kappa^+$  or  $\kappa_1 = \beth_{(2^\kappa)^+} (= \beth_{1,1}(\kappa))$  or even  $\beth_{1,7}(\kappa)$ .

After 4.10 the next natural step is to show that  $\mathfrak{s}_\lambda$  has the better properties dealt with in Chapter III, Chapter IV, see [

Sh:F782]. Note that if we strengthen the assumption on  $\mu$  in §4 (to  $\mu = \mu^{<\lambda}$ ), then it relies on §1 only. Without this we need §2 (hence 5.1(1),(4)).

Originally we have used here categoricity assumptions but lately it seems desirable to use a weaker one: (variants of) solvability. About being solvable, see N§4(B), [Sh 842]. This seems better as it is a candidate for being an “outside” generalization of being superstable (rather than of being categorical).

Here we use solvable when it does not require much change; for more on it see [Sh 842], [Sh:F820] and on material delayed from here see [Sh:F782].

Note we can systematically use  $K^{\text{sc}(\theta)\text{-lin}}$ , say with  $\theta = \aleph_0$  or  $\theta = \text{LS}(\mathfrak{K})$  instead of  $K^{\text{lin}}$ ; see Definition 0.14(8). In several respects this is better, but not enough to make us use it. Also working more it seemed we can get rid of “wide”, “wide over”, see Definition 0.14(1),(2),(3). If instead proving the existence of a good  $\lambda$ -frame it suffices for us to prove the existence of almost good  $\lambda$ -frame, then the assumption on  $\lambda$  can be somewhat weaker (fixed point instead limit of fix points of the sequence of the  $\beth_\alpha$ ’s). In §7 we sometimes give alternative quotations in [Sh 394] but do not rely on it.

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Basic knowledge on infinitary logics is assumed, see e.g. [Di]; though the reader may just read the definition here in N§5 and believe some quoted results.

*0.2 Notation.* Let  $\beth_{0,\alpha}(\lambda) = \beth_\alpha(\lambda) := \lambda + \Sigma\{\beth_\beta(\lambda) : \beta < \alpha\}$ . Let  $\beth_{1,\alpha}(\lambda)$  be defined by induction on  $\alpha$  :  $\beth_{1,0}(\lambda) = \lambda$ , for limit  $\beta$  we let  $\beth_{1,\beta} = \sum_{\gamma < \beta} \beth_{1,\gamma}$  and  $\beth_{1,\beta+1}(\lambda) = \beth_\mu$  where  $\mu = (2^{\beth_{1,\beta}(\lambda)})^+$ .

*0.3 Remark.* 1) For our purpose, usually  $\beth_{1,\beta+1}(\lambda) = \beth_{\delta(\mu)}$  where  $\mu = \beth_{1,\beta}(\lambda)$  suffice, see e.g. II.A§1 in particular on  $\delta(-)$ . Generally  $\mu = (\beth_{1,\beta}(\lambda))^+$  is a more natural definition, but:

- (a) the difference is not significant, e.g. for  $\alpha$  limit we get the same value
- (b) our use of omitting types makes our choice more natural.

2) We do not use but it is natural to define  $\sqsupset_{\gamma+1,0}(\lambda) = \lambda$ ,  $\sqsupset_{\gamma+1,\beta+1}(\lambda) = \sqsupset_{\gamma,\mu}(\lambda)$  with  $\mu = (2^{\sqsupset_{\gamma+1,\beta}(\lambda)})^+$ ,  $\sqsupset_{\gamma+1,\delta}(\lambda) = \sum_{\beta < \delta} \sqsupset_{\gamma+1,\beta}(\lambda)$  and  $\sqsupset_{\delta,0}(\lambda) = \sup\{\sqsupset_{\gamma,0}(\lambda) : \gamma < \delta\} = \lambda$ ,  $\sqsupset_{\delta,\beta+1}(\lambda) = \sqsupset_{\delta,\beta}(\sqsupset_{\delta,\beta}(\lambda))$ ,  $\sqsupset_{\delta,\delta_1} = \sup\{\sqsupset_{\delta,\alpha}(\lambda) : \alpha < \delta_1\}$ ; this is used, e.g. in [Sh:g, ChV].

**0.4 Definition.** Assume  $M$  is a model,  $\tau = \tau_M$  is its vocabulary and  $\Delta$  is a language (or just a set of formulas) in some logic, in the vocabulary  $\tau$ .

For any set  $A \subseteq M$  and set  $\Delta$  of formulas in the vocabulary  $\tau_M$ , let  $\text{Sfr}_\Delta^\alpha(A, M)$  which we call the set of formal  $(\Delta, \alpha)$ -types over  $A$  in  $M$ , be the set of  $p$  such that

- (a)  $p$  a set of formulas of the form  $\varphi(\bar{x}, \bar{a})$  where  $\varphi(\bar{x}, \bar{y}) \in \Delta$ ,  $\bar{x} = \langle x_i : i < \alpha \rangle$  and  $\bar{a} \in {}^{\ell g(\bar{y})}A$
- (b) if  $\Delta$  is closed under negation (which is the case we use here) then for any  $\varphi(\bar{x}, \bar{y}) \in \Delta$  with  $\bar{x}$  as above and  $\bar{a} \in {}^{\ell g(\bar{y})}A$  we have  $\varphi(\bar{x}, \bar{a}) \in p$  or  $\neg\varphi(\bar{x}, \bar{a}) \in p$ .

Recall

**0.5 Definition.** 1) For  $\mathfrak{K}$  an a.e.c. we say  $M \in \mathfrak{K}_\theta$  is a superlimit (model in  $\mathfrak{K}$  or in  $\mathfrak{K}_\theta$ ) when:

- (a)  $M$  is universal
- (b) if  $\delta$  is a limit ordinal  $< \theta^+$  and  $\langle M_\alpha : \alpha \leq \delta \rangle$  is  $\leq_{\mathfrak{K}_\theta}$ -increasing continuous and  $\alpha < \delta \Rightarrow M_\alpha \cong M$  then  $M_\delta \cong M$  (equivalently,  $\mathfrak{K}_\theta^{[M]} = \mathfrak{K} \upharpoonright \{N : N \cong M\}$  is a  $\theta$ -a.e.c.)
- (c) there is  $N$  such that  $M <_{\mathfrak{K}} N \in \mathfrak{K}_\theta$  and  $N$  is isomorphic to  $M$ .

2) We say  $M \in \mathfrak{K}_\theta$  is locally superlimit when we weaken clause (a) to

- (a)<sup>-</sup> if  $N \in \mathfrak{K}_\theta$  is a  $\leq_{\mathfrak{K}}$ -extension of  $M$  then  $N$  can be  $\leq_{\mathfrak{K}}$ -embedded into  $M$ .

3) We say that  $M$  is pseudo superlimit when in part (1) clauses (b),(c) hold (but we omit clause (a)); see 0.6(7) below.

3A) For  $M \in K_\lambda$  let  $\mathfrak{K}_{[M]} = \mathfrak{K}_\lambda^{[M]}$  be  $\mathfrak{K} \upharpoonright \{N : N \cong M\}$ .

4) In (1) we may say globally superlimit.

**0.6 Observation.** Assume ( $\mathfrak{K}$  is an a.e.c. and)  $\mathfrak{K}_\lambda \neq \emptyset$ .

- 1) If  $\mathfrak{K}$  is categorical in  $\lambda$  and there are  $M <_{\mathfrak{K}_\lambda} N$  then every  $M \in \mathfrak{K}_\lambda$  is superlimit.
- 2) If every/some  $M \in \mathfrak{K}_\lambda$  is superlimit then every/some  $M \in K_\lambda$  is locally

superlimit.

3) If every/some  $M \in \mathfrak{K}_\lambda$  is locally superlimit then every/some  $M \in \mathfrak{K}_\lambda$  is pseudo superlimit.

4) If some  $M \in \mathfrak{K}_\lambda$  is superlimit then every locally superlimit  $M' \in \mathfrak{K}_\lambda$  is isomorphic to  $M$ .

5) If  $M$  is superlimit in  $\mathfrak{K}$  then  $M$  is locally superlimit in  $\mathfrak{K}$ . If  $M$  is locally superlimit in  $\mathfrak{K}$ , then  $M$  is pseudo superlimit in  $\mathfrak{K}$ . If  $M$  is locally superlimit in  $\mathfrak{K}_\theta$  then  $\mathfrak{K}_\theta$  has the joint embedding property iff  $M$  is superlimit.

6) In Definition 0.5(1), clause (c) follows from

$$(c)^- \text{ LS}(\mathfrak{K}) \leq \theta \text{ and } K_{\geq \theta^+} \neq \emptyset.$$

7)  $M \in K_\lambda$  is pseudo-superlimit iff  $\mathfrak{K}_{[M]}$  is a  $\lambda$ -a.e.c. and  $\leq_{\mathfrak{K}_{[M]}}$  is not the equality. Also Definition 0.5(3A) is compatible with III. 600-0.33

**0.7 Definition.** For an a.e.c.  $\mathfrak{K}$ , let  $\mathfrak{K}_\mu^{\text{sl}}, \mathfrak{K}_\mu^{\text{ls}}, \mathfrak{K}_\mu^{\text{pl}}$  be the class of  $M \in \mathfrak{K}_\mu$  which are superlimit, locally superlimit, pseudo superlimit respectively with the partial order  $\leq_{\mathfrak{K}_\mu^{\text{sl}}}, \leq_{\mathfrak{K}_\mu^{\text{ls}}}, \leq_{\mathfrak{K}_\mu^{\text{pl}}}$  being  $\leq_{\mathfrak{K}} \upharpoonright K_\mu^{\text{sl}}, \leq_{\mathfrak{K}} \upharpoonright K_\mu^{\text{pl}}$  respectively.

**0.8 Definition.** 1)  $\Phi$  is proper for linear orders when:

- (a) for some vocabulary  $\tau = \tau_\Phi = \tau(\Phi)$ ,  $\Phi$  is an  $\omega$ -sequence, the  $n$ -th element a complete quantifier free  $n$ -type in the vocabulary  $\tau$
- (b) for every linear order  $I$  there is a  $\tau$ -model  $M$  denoted by  $\text{EM}(I, \Phi)$ , generated by  $\{a_t : t \in I\}$  such that  $s \neq t \Rightarrow a_s \neq a_t$  for  $s, t \in I$  and  $\langle a_{t_0}, \dots, a_{t_{n-1}} \rangle$  realizes the quantifier free  $n$ -type from clause (a) whenever  $n < \omega$  and  $t_0 <_I \dots <_I t_{n-1}$ ; so really  $M$  is determined only up to isomorphism but we may ignore this and use  $I_1 \subseteq J_1 \Rightarrow \text{EM}(I_1, \Phi) \subseteq \text{EM}(I_2, \Phi)$ . We call  $\langle a_t : t \in I \rangle$  “the” skeleton of  $M$ ; of course again “the” is an abuse of notation as it is not necessarily unique.

1A) If  $\tau \subseteq \tau(\Phi)$  then we let  $\text{EM}_\tau(I, \Phi)$  be the  $\tau$ -reduct of  $\text{EM}(I, \Phi)$ .

2)  $\Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$  is the class of  $\Phi$  proper for linear orders satisfying clauses (a)( $\alpha$ ), (b), (c) of Claim 0.9(1) below and  $|\tau(\Phi)| \leq \kappa$ . The default value of  $\kappa$  is  $\text{LS}(\mathfrak{K})$  and then we may write  $\Upsilon_\mathfrak{K}^{\text{or}}$  or  $\Upsilon^{\text{or}}[\mathfrak{K}]$  and for simplicity always  $\kappa \geq \text{LS}(\mathfrak{K})$  (and so  $\kappa \geq |\tau_\mathfrak{K}|$ ).

3) We define “ $\Phi$  proper for  $K$ ” similarly when in clause (b) of part (1) we demand  $I \in K$ , so  $K$  is a class of  $\tau_K$ -models, i.e.

- (a)  $\Phi$  is a function, giving for a quantifier free  $n$ -type in  $\tau_K$ , a quantifier free  $n$ -type in  $\tau_\Phi$
- (b)' in clause (b) of part (1), the quantifier free type which  $\langle a_{t_0}, \dots, a_{t_{n-1}} \rangle$  realizes in  $M$  is  $\Phi(\text{tp}_{\text{qf}}(\langle t_0, \dots, t_{n-1} \rangle, \emptyset, M))$  for  $n < \omega, t_0, \dots, t_{n-1} \in I$ .





**0.9 Claim.** 1) Let  $\mathfrak{K}$  be an a.e.c. and  $M \in K$  be of cardinality  $\geq \beth_{1,1}(\text{LS}(\mathfrak{K}))$  recalling we naturally assume  $|\tau_{\mathfrak{K}}| \leq \text{LS}(\mathfrak{K})$  as usual.

Then there is a  $\Phi$  such that  $\Phi$  is proper for linear orders and:

- (a) (α)  $\tau_{\mathfrak{K}} \subseteq \tau_{\Phi}$ ,
- (β)  $|\tau_{\Phi}| = \text{LS}(\mathfrak{K}) + |\tau_{\mathfrak{K}}|$
- (b) for any linear order  $I$  the model  $\text{EM}(I, \Phi)$  has cardinality  $|\tau(\Phi)| + |I|$  and we have  $\text{EM}_{\tau(\mathfrak{K})}(I, \Phi) \in K$
- (c) for any linear orders  $I \subseteq J$  we have  $\text{EM}_{\tau(\mathfrak{K})}(I, \Phi) \leq_{\mathfrak{K}} \text{EM}_{\tau(\mathfrak{K})}(J, \Phi)$
- (d) for every finite linear order  $I$ , the model  $\text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$  can be  $\leq_{\mathfrak{K}}$ -embedded into  $M$ .

2) If we allow  $\text{LS}(\mathfrak{K}) < |\tau_{\mathfrak{K}}|$  and there is  $M \in \mathfrak{K}$  of cardinality  $\geq \beth_{1,1}(\text{LS}(\mathfrak{K}) + |\tau_{\mathfrak{K}}|)$ , then there is  $\Phi \in \Upsilon_{\text{LS}(\mathfrak{K}) + |\tau(\Phi)|}^{\text{or}}[\mathfrak{K}]$  such that  $\text{EM}(I, \Phi)$  has cardinality  $\leq \text{LS}(\mathfrak{K})$  for  $I$  finite. Hence  $\mathcal{E}$  has  $\leq 2^{\text{LS}(\mathfrak{K})}$  equivalence classes where  $\mathcal{E} = \{(P_1, P_2) : P_1, P_2 \in \tau_{\Phi} \text{ and } P_1^{\text{EM}(I, \Phi)} = P_2^{\text{EM}(I, \Phi)} \text{ for every linear order } I\}$ .

3) Actually having a model of cardinality  $\geq \beth_{\alpha}$  for every  $\alpha < (2^{\text{LS}(\mathfrak{K}) + |\tau(\mathfrak{K})|})^+$  suffice (in part (2)).

*Proof.* Follows from the existence of a representation of  $\mathfrak{K}$  as a  $\text{PC}_{\mu, 2^{\mu}}$ -class when  $\mu = \text{LS}(\mathfrak{K}) + |\tau(\mathfrak{K})|$  in I.  88r-1.4  (3), (4), (5) and I.  88r-1.8  (or see [Sh 394, 0.6]). □<sub>0.9</sub>

**0.10 Remark.** Note that some of the definitions and claims below will be used only in remarks:  $K_{\theta}^{\text{sc}(\kappa)}$  from 0.14(8), in 1.7; and some only in §6, §7 (and part of §5 needed for it):  $\Upsilon_{\kappa}^{\text{lin}}[2]$  from 0.11(5) (and even less  $\Upsilon_{\kappa}^{\text{lin}}[\alpha(*)]$  from Definition 0.14(9)). Also the use of  $\leq_{\kappa}^{\otimes}, \leq_{\kappa}^{\text{ie}}, \leq_{\kappa}^{\oplus}$  is marginal.

**0.11 Definition.** We define partial orders  $\leq_{\kappa}^{\oplus}, \leq_{\kappa}^{\text{ie}}$  and  $\leq_{\kappa}^{\otimes}$  on  $\Upsilon_{\kappa}^{\text{or}}[\mathfrak{K}]$  (for  $\kappa \geq \text{LS}(\mathfrak{K})$ ) as follows:

- 1)  $\Psi_1 \leq_{\kappa}^{\oplus} \Psi_2$  iff  $\tau(\Psi_1) \subseteq \tau(\Psi_2)$  and  $\text{EM}_{\tau(\mathfrak{K})}(I, \Psi_1) \leq_{\mathfrak{K}} \text{EM}_{\tau(\mathfrak{K})}(I, \Psi_2)$  and  $\text{EM}(I, \Psi_1) = \text{EM}_{\tau(\Psi_1)}(I, \Psi_1) \subseteq \text{EM}_{\tau(\Psi_2)}(I, \Psi_2)$  for any linear order  $I$ .

Again for  $\kappa = \text{LS}(\mathfrak{K})$  we may drop the  $\kappa$ .

- 2) For  $\Phi_1, \Phi_2 \in \Upsilon_{\kappa}^{\text{or}}[\mathfrak{K}]$ , we say  $\Phi_2$  is an inessential extension of  $\Phi_1$  and write  $\Phi_1 \leq_{\kappa}^{\text{ie}} \Phi_2$  iff  $\Phi_1 \leq_{\kappa}^{\oplus} \Phi_2$  and for every linear order  $I$ , we have (note: there may be more function symbols in  $\tau(\Phi_2)$ !)

$$\text{EM}_{\tau(\mathfrak{K})}(I, \Phi_1) = \text{EM}_{\tau(\mathfrak{K})}(I, \Phi_2).$$

3) Let  $\Upsilon_\kappa^{\text{lin}}$  be the class of  $\Psi$  proper for linear order and (producing a linear order extending the original one, i.e.) such that:

- (a)  $\tau(\Psi)$  has cardinality  $\leq \kappa$  and the two-place predicate  $<$  belongs to  $\tau(\Psi)$
- (b)  $\text{EM}_{\{<\}}(I, \Psi)$  is a linear order which is an extension of  $I$  in the sense that  $\text{EM}(I, \Phi) \models "a_s < a_t"$  iff  $I \models "s < t"$ ; in fact we usually stipulate  $[t \in I \Rightarrow a_t = t]$ .

4)  $\Phi_1 \leq_\kappa^\otimes \Phi_2$  iff there is  $\Psi$  such that

- (a)  $\Psi \in \Upsilon_\kappa^{\text{lin}}$
- (b)  $\Phi_\ell \in \Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$  for  $\ell = 1, 2$
- (c)  $\Phi'_2 \leq_\kappa^{\text{ie}} \Phi_2$  where  $\Phi'_2 = \Psi \circ \Phi_1$ , i.e. for every linear order  $I$  we have

$$\text{EM}(I, \Phi'_2) = \text{EM}(\text{EM}_{\{<\}}(I, \Psi), \Phi_1).$$

5)  $\Upsilon_\kappa^{\text{lin}}[2]$  is the class of  $\Psi$  proper for  $K_{\tau_2^*}^{\text{lin}}$  and producing structures from  $K_{\tau_2^*}^{\text{lin}}$  extending the originals, i.e.

- (a)  $\tau_2^* = \{<, P_0, P_1\}$  where  $P_0, P_1$  are unary predicates,  $<$  a binary predicate
- (b)  $K_{\tau_2^*}^{\text{lin}} = \{M : M \text{ a } \tau_2^*\text{-model, } <^M \text{ a linear order, } \langle P_0^M, P_1^M \rangle \text{ a partition of } M\}$
- (c) the two-place predicate  $<$  and the one place predicates  $P_0, P_1$  belong to  $\tau(\Psi)$
- (d) if  $I \in K_{\tau_2^*}^{\text{lin}}$  then  $M = \text{EM}_{\tau_2^*}(I, \Phi)$  belongs to  $K_{\tau_2^*}^{\text{lin}}$  and  $<^M$  is a linear order and  $I \models s < t \Rightarrow M \models a_s < a_t$  and  $t \in P_\ell^I \Rightarrow a_t \in P_\ell^M$ .

6) Similarly  $\Upsilon_\kappa^{\text{lin}}[\alpha(*)]$  using  $K_{\tau_{\alpha(*)}^*}^{\text{lin}}$  (see below in 0.14(9)).

**0.12 Claim.** Assume  $\Phi \in \Upsilon_{\mathfrak{K}}^{\text{or}}$ .

1) If  $\pi$  is an isomorphism from the linear order  $I_1$  onto the linear order  $I_2$  then it induces a unique isomorphism  $\hat{\pi}$  from  $M_1 = \text{EM}(I_1, \Phi)$  onto  $M_2 = \text{EM}(I_2, \Phi)$  such that:

- (a)  $\hat{\pi}(a_t) = a_{\pi(t)}$  for  $t \in I$
- (b)  $\hat{\pi}(\sigma^{M_1}(a_{t_0}, \dots, a_{t_{n-1}})) = \sigma^{M_2}(a_{\pi(t_0)}, \dots, a_{\pi(t_{n-1})})$  where  $\sigma(x_0, \dots, x_{n-1})$  is a  $\tau_\Phi$ -term and  $t_0, \dots, t_{n-1} \in I_1$ .

2) If  $\pi$  is an automorphism of the linear order  $I$  then it induces a unique automorphism  $\hat{\pi}$  of  $\text{EM}(I, \Phi)$  (as above with  $I_1 = I = I_2$ ).

*0.13 Remark.* 1) So in 0.11(2) we allow further expansion by functions definable from earlier ones (composition or even definition by cases), as long as the number is  $\leq \kappa$ .

2) Of course, in 0.12 is true for trivial  $\mathfrak{K}$ .

So we may be interested in some classes of linear orders; below 0.14(1) is used much more than the others and also 0.14(5),(6) are used not so few times, in particular parts (8),(9) are not used till §5.

**0.14 Definition.** 1) A linear order  $I$  is  $\kappa$ -wide when for every  $\theta < \kappa$  there is a monotonic sequence of length  $\theta^+$  in  $I$ .

2) A linear order  $I$  is  $\kappa$ -wider if  $|I| \geq \beth_{1,1}(\kappa)$ .

3)  $I_2$  is  $\kappa$ -wide over  $I_1$  if  $I_1 \subseteq I_2$  and for every  $\theta < \kappa$  there is a convex subset of  $I_2$  disjoint to  $I_1$  which is  $\theta^+$ -wide. We say “ $I_2$  is wide over  $I_1$ ” if “ $I_2$  is  $|I_1|$ -wide over  $I_2$ ”.

4)  $K^{\text{lin}}_\lambda$  is the class of linear orders [of cardinality  $\lambda$ ].

5) Let  $K^{\text{flin}}$  be the class of infinite linear order  $I$  such that every interval has cardinality  $|I|$  and is with neither first nor last elements.

6) Let the two-place relation  $\leq_{K^{\text{flin}}}$  on  $K^{\text{flin}}$  be defined by:  $I \leq_{K^{\text{flin}}} J$  iff  $I, J \in K^{\text{flin}}$  and  $I \subseteq J$  and either  $I = J$  or  $J \setminus I$  is a dense subset of  $J$  and for every  $t \in J \setminus I$ ,  $I$  can be embedded into  $J \upharpoonright \{s \in J \setminus I : (\forall r \in I)(s <_J r \equiv t <_J r)\}$ .

6A) Let the two-place relation  $\leq^*_{K^{\text{flin}}}$  on  $K^{\text{lin}}$  be defined similarly omitting “ $I \in K^{\text{flin}}$ ” (but not  $J \in K^{\text{flin}}$ ).

7)  $K^{\text{flin}}_\theta = \{I \in K^{\text{flin}} : |I| = \theta\}$  and  $\leq_{K^{\text{flin}}_\theta} = \leq_{K^{\text{flin}}} \upharpoonright K^{\text{flin}}_\theta$ .

8)  $K^{\text{sc}(\kappa)-\text{lin}}_\theta$  is the class of linear orders of cardinality  $\theta$  which are the union of  $\leq \kappa$  scattered linear orders (recalling  $I$  is scattered when there is no  $J \subseteq I$  isomorphic to the rationals). If  $\kappa = \aleph_0$  we may omit it (i.e. write  $K^{\text{sc-lin}}_\theta$ ).

9) Let  $\tau^*_{\alpha(*)} = \{<\} \cup \{P_i : i < \alpha(*)\}$ ,  $P_i$  a monadic predicate,  $K^{\text{lin}}_{\tau^*_{\alpha(*)}} = \{I : I \text{ a } \tau^*_{\alpha(*)}\text{-model, } <^I \text{ a linear order and } \langle P_i^I : i < \alpha(*) \rangle \text{ a partition of } I\}$ . If  $\alpha(*) = 1$  we may omit  $P_0^I$ , so  $I$  is a linear order, so any ordinal can be treated as a member of  $K^{\text{lin}}_{\tau^*_1}$ .

*0.15 Observation.* 1) If  $|I| > 2^\theta$  then  $I$  is  $\theta^+$ -wide.

2) If  $|I| \geq \lambda$  and  $\lambda$  is a strong limit cardinal then  $I$  is  $\lambda$ -wide.

3)  $(K^{\text{flin}}_\theta, \leq_{K^{\text{flin}}_\theta})$  almost is a  $\theta$ -a.e.c., only smoothness may fail.

- 4) If  $I_1 \in K^{\text{lin}}$  then for some  $I_2 \in K^{\text{flin}}$  we have:  $|I_2| = |I_1| + \aleph_0$  and  $I_1 \leq_{K^{\text{flin}}}^* I_2$ ; and  $(\forall I_0)[I_0 \subseteq I_1 \wedge I_0 \in K^{\text{flin}} \Rightarrow I_0 \leq_{K^{\text{flin}}} I_2]$ .
- 5) If  $I_1$  is  $\kappa$ -wide and  $I_1 <_{K^{\text{flin}}} I_2$  then  $I_2$  is  $\kappa$ -wide over  $I_2$ .

*Remark.* If in the definition of  $\leq_{K^{\text{flin}}}$  in 0.14(6) we can add “ $(\forall t \in I)(\exists t' \in J)[t' <_J t \wedge (\forall s \in I)(s <_I t \rightarrow s <_J t')]$ ” (and its dual, i.e. inverting the order). So we can strengthen 0.14(6) by the demand above.

*Proof.* 1) By Erdős-Rado Theorem, i.e., by  $(2^\theta)^+ \rightarrow (\theta^+)_2^2$ .

2) Follows by part (1).

3),4),5) Easy. □<sub>0.15</sub>

**0.16 Claim.** 1)  $(\Upsilon_{\kappa[\mathfrak{K}]}^{\text{or}}, \leq_{\kappa}^{\otimes})$ ,  $(\Upsilon_{\kappa}^{\text{or}}[\mathfrak{K}], <_{\kappa}^{\text{ie}})$  and  $(\Upsilon_{\kappa[\mathfrak{K}]}^{\text{or}}, \leq^{\oplus})$  are partial orders (and  $\leq_{\kappa}^{\otimes}, \leq_{\kappa}^{\text{ie}} \subseteq \leq_{\kappa}^{\oplus}$ ).

2) If  $\Phi_i \in \Upsilon_{\kappa}^{\text{or}}[\mathfrak{K}]$  and the sequence  $\langle \Phi_i : i < \delta \rangle$  is a  $\leq_{\kappa}^{\otimes}$ -increasing sequence,  $\delta < \kappa^+$ , then it has a  $<_{\kappa}^{\otimes}$ -l.u.b.  $\Phi \in \Upsilon_{\kappa}^{\text{or}}[\mathfrak{K}]$ , and  $\text{EM}(I, \Phi) = \bigcup_{i < \delta} \text{EM}(I, \Phi_i)$  for

every linear order  $I$ , i.e.  $\tau(\Phi) = \cup\{\tau(\Phi_i) : i < \delta\}$  and for every  $j < \delta$  we have  $\text{EM}_{\tau(\Phi_j)}(I, \Phi) = \cup\{\text{EM}_{\tau(\Phi_i)}(I, \Phi) : i \in [j, \delta)\}$ .

3) Similarly for  $<_{\kappa}^{\oplus}$  and  $\leq_{\kappa}^{\text{ie}}$ .

4) If  $\Phi \in \Upsilon_{\kappa}^{\text{lin}}$  and  $I \in K^{\text{lin}}$  then  $I \subseteq \text{EM}_{\{<\}}(I, \Phi)$  as linear orders stipulating (as in 0.11(3)) that  $a_t = t$ .

*Proof.* Easy. □<sub>0.16</sub>

Recall various well known facts on  $\mathbb{L}_{\infty, \theta}$ .

**0.17 Claim.** 1) If  $M, N$  are  $\tau$ -models of cardinality  $\lambda$ ,  $\text{cf}(\lambda) = \aleph_0$  and  $M \equiv_{\mathbb{L}_{\infty, \lambda}} N$  then  $M \cong N$ .

2) If  $M, N$  are  $\tau$ -models then  $M \equiv_{\mathbb{L}_{\infty, \theta}} N$  iff there is  $\mathcal{F}$  such that

- ⊛ (a)(α) each  $f \in \mathcal{F}$  is a partial isomorphism from  $M$  to  $N$
- (β)  $\mathcal{F} \neq \emptyset$
- (γ) if  $f \in \mathcal{F}$  and  $A \subseteq \text{Dom}(f)$  then  $f \upharpoonright A \in \mathcal{F}$
- (b) if  $f \in \mathcal{F}$ ,  $A \in [M]^{<\theta}$  and  $B \in [N]^{<\theta}$  then for some  $g \in \mathcal{F}$  we have  $f \subseteq g$ ,  $A \subseteq \text{Dom}(g)$ ,  $B \subseteq \text{Rang}(g)$ .



2A) If  $M \subseteq N$  are  $\tau$ -models, then  $M \prec_{\mathbb{L}_{\infty, \theta}} N$  iff for some  $\mathcal{F}$  clauses  $\otimes(a), (b)$  hold together with

(c) if  $A \in [M]^{<\theta}$  then for some  $f \in \mathcal{F}$  we have  $\text{id}_A \subseteq f$ .

2B) In part (2) (and part (2A)), we can omit subclause  $(\gamma)$  of clause (a), and if  $\mathcal{F}$  satisfies  $(a)(\alpha), (\beta) + (b)$  (and (c)), then also  $\mathcal{F}' = \{f \upharpoonright A : f \in \mathcal{F} \text{ and } A \subseteq \text{Dom}(f)\}$  satisfies the demands.

2C) Let  $M, N$  be  $\tau$ -models and define  $\mathcal{F} = \{f : \text{for some } \bar{a} \in {}^{\theta>}M, f \text{ is a function from } \text{Rang}(\bar{a}) \text{ to } N \text{ such that } (M, \bar{a}) \equiv_{\mathbb{L}_{\infty, \theta}} (N, f(\bar{a}))\}$  then  $M \equiv_{\mathbb{L}_{\infty, \theta}} N$  iff  $\mathcal{F} \neq \emptyset$  iff  $\mathcal{F}$  satisfies clauses (a), (b) of  $\otimes$ .

3) If  $M$  is a  $\tau$ -model,  $\theta = \text{cf}(\theta)$  and  $\mu = \|M\|^{<\theta}$  then for some  $\gamma < \mu^+$  and  $\Delta \subseteq \mathbb{L}_{\mu^+, \theta}(\tau)$  of cardinality  $\leq \mu$  such that each  $\varphi(\bar{x}) \in \Delta$  is of quantifier depth  $< \gamma$ , we have

(a) for  $\bar{a}, \bar{b} \in {}^{\theta>}M$  we have  $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty, \theta}} (M, \bar{b})$  iff  $\text{tp}_{\Delta}(\bar{a}, \emptyset, M) = \text{tp}_{\Delta}(\bar{b}, \emptyset, M)$

(b) for any  $\tau$ -model  $N$  we have  $N \equiv_{\mathbb{L}_{\infty, \theta}} M$  iff  $\{\text{tp}_{\Delta}(\bar{a}, \emptyset, N) : \bar{a} \in {}^{\theta>}N\} = \{\text{tp}_{\Delta}(\bar{a}, \emptyset, M) : \bar{a} \in {}^{\theta>}M\}$ .

4) Assume  $\chi > \mu = \mu^{<\kappa}$  and  $x \in \mathcal{H}(\chi)$ . There is  $\mathfrak{B}$  such that (in fact clauses (d)-(g) follow from clauses (a), (b), (c))

(a)  $\mathfrak{B} \prec (\mathcal{H}(\chi), \in)$  has cardinality  $\mu$ ,

(b)  $\mu + 1 \subseteq \mathfrak{B}$  and  $[\mathfrak{B}]^{<\kappa} \subseteq \mathfrak{B}$  and  $x \in \mathfrak{B}$

(c)  $\mathfrak{B} \prec_{\mathbb{L}_{\kappa, \kappa}} (\mathcal{H}(\chi), \in)$

(d) if  $\mathfrak{K}$  is an a.e.c. with  $\text{LS}(\mathfrak{K}) + |\tau(\mathfrak{K})| \leq \mu$  and  $\mathfrak{K} \in \mathfrak{B}$  (which means  $\{(M, N) : M \leq_{\mathfrak{K}} N \text{ has universes } \subseteq \text{LS}(\mathfrak{K})\} \in \mathfrak{B}$ ) then

( $\alpha$ )  $M \in \mathfrak{K} \cap \mathfrak{B} \Rightarrow M \upharpoonright \mathfrak{B} := M \upharpoonright (\mathfrak{B} \cap M) \leq_{\mathfrak{K}} M$

( $\beta$ ) if  $M \leq_{\mathfrak{K}} N$  belongs to  $\mathfrak{B}$  then  $M \upharpoonright \mathfrak{B} \leq_{\mathfrak{K}} N \upharpoonright \mathfrak{B}$

(e) if  $\mathfrak{K}$  is as in (d),  $\Phi \in \Upsilon_{\leq \mu}^{\text{or}}[\mathfrak{K}] \cap \mathfrak{B}$  and  $I \in \mathfrak{B}$  is a linear order and so  $M = \text{EM}(I, \Phi) \in \mathfrak{B}$  then  $I' = I \upharpoonright \mathfrak{B} \subseteq I$  and  $M \upharpoonright \mathfrak{B} = \text{EM}(I', \Phi)$  so  $(M \upharpoonright \tau(\mathfrak{K})) \upharpoonright \mathfrak{B} = \text{EM}_{\tau(\mathfrak{K})}(I', \Phi) \leq_{\mathfrak{K}} M \upharpoonright \tau(\mathfrak{K})$

(f) if  $|\tau| \leq \mu$ ,  $\tau \in \mathfrak{B}$  and  $M, N \in \mathfrak{B}$  are  $\tau$ -models, then

( $\alpha$ )  $M \upharpoonright \mathfrak{B} \prec_{\mathbb{L}_{\kappa, \kappa}[\tau]} M$

( $\beta$ )  $M \not\equiv_{\mathbb{L}_{\infty, \kappa}[\tau]} N$  iff  $(M \upharpoonright \mathfrak{B}) \not\equiv_{\mathbb{L}_{\infty, \kappa}[\tau]} (N \upharpoonright \mathfrak{B})$

( $\gamma$ ) if  $M \subseteq N$  then  $(M \prec_{\mathbb{L}_{\infty, \kappa}(\tau)} N)$  iff  $(M \upharpoonright \mathfrak{B}) \prec_{\mathbb{L}_{\infty, \kappa}(\tau)} (N \upharpoonright \mathfrak{B})$ ; this applies also to  $(M, \bar{a}), (N, \bar{a})$  for  $\bar{a} \in {}^{\kappa>}M$

(g) if  $I \in K^{\text{flin}}$  then  $I_1 \cap \mathfrak{B} \in K^{\text{flin}}$  and if  $I_1 <_{K^{\text{flin}}}^* I_2$  then  $(I_1 \cap \mathfrak{B}) <_{K^{\text{flin}}}^* (I_2 \cap \mathfrak{B})$ .

*Proof.* 1)-3) and 4)(a),(b),(c) Well known, e.g. see [Di].

4) Clauses (d),(e),(f): as in 0.9(1), i.e. by absoluteness. Also clause (g) should be clear.  $\square_{0.17}$

0.18 Remark. 1) We will be able to add, in 0.17(4):

(g) if  $\mathfrak{K}$  is as in clause (d) and  $\tau = \tau_{\mathfrak{K}}$  then in clause (f) we can replace  $\mathbb{L}_{\infty, \kappa}(\tau)$  by  $\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]$  and  $\mathbb{L}_{\kappa, \kappa}(\tau)$  by  $\mathbb{L}_{\kappa, \kappa}[\mathfrak{K}]$ , see Definition 1.9 and Fact 1.10(5).

2) We use part (4) in 1.26(3).

**0.19 Definition.** For a model  $M$  and for a set  $\Delta$  of formulas in the vocabulary of  $M$ ,  $\bar{x} = \langle x_i : i < \alpha \rangle$ ,  $A \subseteq M$  and  $\bar{a} \in {}^\alpha M$  let the  $\Delta$ -type of  $\bar{a}$  over  $A$  in  $M$  be  $\text{tp}_\Delta(\bar{a}, A, M) = \{\varphi(\bar{x}, \bar{b}) : M \models \varphi[\bar{a}, \bar{b}] \text{ where } \varphi = \varphi(\bar{x}, \bar{y}) \in \Delta \text{ and } \bar{b} \in {}^{\ell g(\bar{y})} A\}$ .

## §1 AMALGAMATION IN $K_\lambda^*$

Our aim is to investigate what is implied by 1.3 below but instead of assuming it we shall shortly assume only some of its consequences. For our purpose here, for  $\theta \in [\text{LS}(\mathfrak{K}), \lambda)$ ,  $\lambda = \beth_\lambda$  it does not really matter if we use  $\kappa = \beth_{1,1}(\theta)$  or  $\kappa = \beth_{1,1}(\beth_n(\theta))$  or  $\beth_{1,n}(\theta)$ , as we are trying to analyze models in  $K_\lambda$ .

1.1 Remark. 1) We can in our claims use only  $\Phi \in \Upsilon_{\mathfrak{K}}^{\text{or}} = \Upsilon_{\text{LS}(\mathfrak{K})}^{\text{or}}[\mathfrak{K}]$  because for every  $\theta \geq \text{LS}(\mathfrak{K})$  we can replace  $\mathfrak{K}$  by  $\mathfrak{K}_{\geq \theta}$  as  $\text{LS}(\mathfrak{K}_{\geq \theta}) = \theta$  when  $\mathfrak{K}_{\geq \theta} \neq \emptyset$ , of course.

2) As usual we assume  $|\tau_{\mathfrak{K}}| \leq \text{LS}(\mathfrak{K})$  just for convenience, otherwise we should just replace  $\text{LS}(\mathfrak{K})$  by  $\text{LS}(\mathfrak{K}) + |\tau_{\mathfrak{K}}|$ .

### 1.2 Hypothesis.

- (a)  $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$  is an a.e.c. with vocabulary  $\tau = \tau(\mathfrak{K})$  (and we can assume  $|\tau| \leq \text{LS}(\mathfrak{K})$  for notational simplicity)
- (b)  $\mathfrak{K}$  has arbitrarily large models (equivalently has a model of cardinality  $\geq \beth_{1,1}(\text{LS}(\mathfrak{K}))$ ), not used, e.g. in 1.10, 1.11 but from 1.12 on it is used extensively.

**1.3 Definition.** We say  $(\mu, \lambda)$  or really  $(\mu, \lambda, \Phi)$  is a weak/strong/pseudo  $\mathfrak{K}$ -candidate when (weak is the default value):

- (a)  $\mu > \lambda = \beth_\lambda > \text{LS}(\mathfrak{K})$  (e.g. the first beth fix point  $> \text{LS}(\mathfrak{K})$ , see 3.5; in the main case  $\lambda$  has cofinality  $\aleph_0$ )
- (b)  $\mathfrak{K}$  categorical in  $\mu$  and  $\Phi \in \Upsilon_{\mathfrak{K}}^{\text{or}}$   
or just
- (b)<sup>-</sup>  $\mathfrak{K}$  is weakly/strongly/pseudo solvable in  $\mu$  and  $\Phi \in \Upsilon_{\mathfrak{K}}^{\text{or}}$  witnesses it; see below.

**1.4 Definition.** 1) We say  $\mathfrak{K}$  is weakly  $(\mu, \kappa)$ -solvable when  $\mu \geq \kappa \geq \text{LS}(\mathfrak{K})$  and there is  $\Phi \in \Upsilon_{\kappa}^{\text{or}}[\mathfrak{K}]$  witnessing it, which means that  $\Phi \in \Upsilon_{\kappa}^{\text{or}}[\mathfrak{K}]$  and  $\text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$  is a locally superlimit member of  $\mathfrak{K}_{\mu}$  for every linear order  $I$  of cardinality  $\mu$ . We may say  $(\mathfrak{K}, \Phi)$  is weakly  $(\mu, \kappa)$ -solvable and we may say  $\Phi$  witness that  $\mathfrak{K}$  is weakly  $(\mu, \kappa)$ -solvable.

If  $\kappa = \text{LS}(\mathfrak{K})$  we may omit it, saying  $\mathfrak{K}$  or  $(\mathfrak{K}, \Phi)$  is weakly  $\mu$ -solvable in  $\mu$ .

2)  $\mathfrak{K}$  is strongly  $(\mu, \kappa)$ -solvable when  $\mu \geq \kappa \geq \text{LS}(\mathfrak{K})$  and some  $\Phi \in \Upsilon_{\kappa}^{\text{or}}[\mathfrak{K}]$  witness it which means that if  $I \in K_{\lambda}^{\text{lin}}$  then  $\text{EM}_{\tau[\mathfrak{K}]}(I, \Phi)$  is superlimit (for  $\mathfrak{K}_{\lambda}$ ). We use the conventions from part (1).

3) We say  $\mathfrak{K}$  is pseudo  $(\mu, \kappa)$ -solvable when  $\mu \geq \kappa \geq \text{LS}(\mathfrak{K})$  and there is  $\Phi \in \Upsilon_{\kappa}^{\text{or}}[\mathfrak{K}]$  witnessing it which means that for some  $\mu$ -a.e.c.  $\mathfrak{K}'$  with no  $\leq_{\mathfrak{K}'}$ -maximal member, we have  $M \in \mathfrak{K}'$  iff  $M \cong \text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$  for some  $I \in K_{\mu}^{\text{lin}}$  iff  $M \cong \text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$  for every  $I \in K_{\mu}^{\text{lin}}$ . We use the conventions from part (1).

4) Let  $(\mu, \kappa)$ -solvable mean weakly  $(\mu, \kappa)$ -solvable, etc., (including 1.3)

**1.5 Claim.** 1) In Definition 1.3, clause (b) implies clause (b)<sup>-</sup>. Also in Definition 1.4 “ $\mathfrak{K}$  is strongly  $(\mu, \kappa)$ -solvable” implies “ $\mathfrak{K}$  is weakly  $(\mu, \kappa)$ -solvable” which implies “ $\mathfrak{K}$  is pseudo  $(\mu, \kappa)$ -solvable”. Similarly for  $(\mathfrak{K}, \Phi)$ .

2) Assume  $\Phi \in \Upsilon_{\kappa}^{\text{or}}[\mathfrak{K}]$ ; if clause (b)<sup>-</sup> of 1.3 or just  $\dot{I}(\mu, \mathfrak{K}) < 2^{\mu}$ , or just  $2^{\mu} > \dot{I}(\mu, \{\text{EM}_{\tau(\mathfrak{K})}(I, \Phi) : I \in K_{\mu}^{\text{lin}}\})$  for some  $\mu$  satisfying  $\text{LS}(\mathfrak{K}) < \kappa^{+} < \mu$  then we can deduce that

(\*)  $\Phi$ , really  $(\mathfrak{K}, \Phi)$  has the  $\kappa$ -non-order property, where the  $\kappa$ -non-order property means that:

if  $I$  is a linear order of cardinality  $\kappa$ ,  $\bar{t}^1, \bar{t}^2 \in {}^{\kappa}I$  form a  $\Delta$ -system pair (see below) and  $\langle \sigma_i(\bar{x}) : i < \kappa \rangle$  lists the  $\tau(\Phi)$ -terms (with the sequence  $\bar{x}$  of variables being  $\langle x_i : i < \kappa \rangle$ ) and  $\langle a_t : t \in I \rangle$  is “the” indiscernible sequence generating  $\text{EM}(I, \Phi)$  (i.e. as usual “ $\langle a_t : t \in I \rangle$ ” is “the” skeleton of  $\text{EM}(I, \Phi)$ , so generating it, see Definition 0.8) then for some  $J \supseteq I$  there is an automorphism of  $\text{EM}_{\tau(\mathfrak{K})}(J, \Phi)$  which exchanges  $\langle \sigma_i(\langle a_{t_i^1} : i < \kappa \rangle) : i < \kappa \rangle$  and

$$\langle \sigma_i(\langle a_{t_i^2} : i < \kappa \rangle) : i < \kappa \rangle.$$

where

- ⊠  $\bar{t}^1, \bar{t}^2 \in {}^\alpha I$  is a  $\Delta$ -system pair when for some  $J \supseteq I$  there are  $\bar{t}^\zeta \in {}^\alpha J$  for  $\zeta \in \kappa \setminus \{1, 2\}$  such that  $\langle \bar{t}^\alpha : \alpha < \kappa \rangle$  is an indiscernible sequence for quantifier free formulas in the linear order  $J$ .

*Proof.* 1) The first sentence holds by Claim 0.9(1) and Definition 0.8 (and Claim 0.6). The second and third sentences follows by 0.6.

2) Otherwise we get a contradiction by [Sh 300, Ch.III] or better [Sh:e, III].  $\square_{1.4}$

**1.6 Definition.** 1) If  $\mathcal{M}'$  is a class of linear orders and  $\Phi \in \Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$  then we let  $K[\mathcal{M}', \Phi] = \{\text{EM}_{\tau(\mathfrak{K})}(I, \Phi) : I \in \mathcal{M}'\}$ .

2) Let  $K_\theta^{u(\kappa)\text{-lin}}$  be the class of linear orders  $I$  of cardinality  $\theta$  such that for some scattered<sup>1</sup> linear order  $J$  and  $\Phi$  proper for  $K^{\text{lin}}$  such that  $<$  belongs to  $\tau_\Phi$ ,  $|\tau_\Phi| \leq \kappa$  we have  $I$  is embeddable into  $\text{EM}_{\{<\}}(J, \Phi)$ . If we omit  $\kappa$  we mean  $\text{LS}(\mathfrak{K})$ . If  $\kappa = \aleph_0$  we may omit it.

**1.7 Remark.** 1) Note that in Definition 1.4(1) we can restrict ourselves to  $I \in K_\lambda^{\text{sc}(\theta)\text{-lin}}$ , see 0.14(8) and even  $I \in K^{u(\theta)\text{-lin}}$  see 1.6(2), i.e., assume  $2^\mu > \dot{I}(\mu, K[\mathcal{M}', \Phi])$ , for  $\mathcal{M}' = K_\lambda^{\text{sc}(\theta)\text{-lin}}$  or  $\mathcal{M}' = K_\lambda^{u(\theta)\text{-lin}}$  and restrict the conclusion (\*) to  $I \in K^{\text{sc}(\theta)\text{-lin}}$ . A gain is that, if  $\lambda > \theta$ , every  $I \in K_\lambda^{\text{sc}(\theta)\text{-lin}}$  is  $\lambda$ -wide so later  $K^* = K^{**}$ , and being solvable is a weaker demand. But it is less natural. Anyhow we presently do not deal with this.

1A) Note that  $K_\lambda^{\text{sc}(\theta)\text{-lin}} \supseteq K_\lambda^{u(\theta)\text{-lin}}$ .

2) An aim of 1.8 below is to show that: by changing  $\Phi$  instead of assuming  $I_1 \subset I_2 \wedge (I_2 \text{ is } \kappa\text{-wide over } I_1)$  it suffices to assume  $I_1 \subset I_2 \wedge (I_2 \text{ is } \kappa\text{-wide})$ .

**1.8 Claim.** For every  $\Phi_1 \in \Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$  there is  $\Phi_2$  such that

- (a)  $\Phi_2 \in \Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$  and if  $\Phi_1$  witnesses  $\mathfrak{K}$  is weakly/strongly/pseudo  $(\lambda, \kappa)$ -solvable then so does  $\Phi_2$
- (b)  $\tau_{\Phi_1} \subseteq \tau_{\Phi_2}$  and  $|\tau_{\Phi_2}| = |\tau_{\Phi_1}| + \aleph_0$
- (c) for any  $I_2 \in K^{\text{lin}}$  there are  $I_1$  and  $h$  such that:
  - (α)  $I_1 \in K^{\text{lin}}$  and even  $I_1 \in K^{\text{flin}}$ , see 0.14(5)

---

<sup>1</sup>i.e. one into which the rational order cannot be embedded

- ( $\beta$ )  $h$  is an embedding of  $I_2$  into  $I_1$
- ( $\gamma$ ) there is an isomorphism  $f$  from  $\text{EM}_{\tau(\Phi_1)}(I_2, \Phi_2)$  onto  $\text{EM}(I_1, \Phi_1)$  such that  $f(a_t) = a_{h(t)}$  for  $t \in I_2$
- ( $\delta$ ) if  $J_1 = I_1 \upharpoonright \text{Rang}(h)$  and we let  $\mathcal{E} = \{(t_1, t_2) : t_1, t_2 \in I_1 \setminus J_1 \text{ and } (\forall s \in J_1)(s < t_1 \equiv s < t_2)\}$  then:  $\mathcal{E}$  is an equivalence relation and each equivalence class has  $\geq |I_2|$  members and  $J_1 \leq_{K^{\text{fin}}} I_1$ , see 0.14(6)
- ( $\varepsilon$ ) [not used] if  $\emptyset \neq J_2 \subseteq I_2$ ,  $J_1 = \{t \in I_1 : \text{for some } \tau(\Phi_2)\text{-term } \sigma(x_0, \dots, x_{n-1}) \text{ and some } t_0, \dots, t_{n-1} \in J_2 \text{ we have } f^{-1}(a_t) = \sigma^{\text{EM}(I_2, \Phi_2)}(a_{t_0}, \dots, a_{t_{n-1}})\}$  and  $J'_1 \subseteq \text{Rang}(h) \setminus J_1$  and  $t \in J'_1$  then  $\{s \in t/\mathcal{E} : f^{-1}(a_s) \text{ belongs to the Skolem hull of } \{f^{-1}(a_r) : r \in J'_1\} \text{ in } \text{EM}(I_2, \Phi)\}$  has cardinality  $\geq |J'_1|$  and  $J'_1$  and its inverse can be embedded into it; in fact,  $I_1$  and its inverse are embeddable into any interval of  $I_2$ .

*Remark.* 1) We can express it by  $\leq_{\kappa}^{\otimes}$ , see 0.11(4). So for some  $\Psi$  proper for linear orders such that  $\tau_{\Psi}$  is countable, the two-place predicate  $<$  belongs to  $\tau_{\Psi}$  and above  $\text{EM}_{\{<\}}(I_2, \Psi)$  is  $I_1$ .

2) In fact,  $J_2 \subset I_2 \Rightarrow \text{EM}_{\{<\}}(J_2, \Psi) <_{K^{\text{fin}}} \text{EM}_{\{<\}}(I_2, \Psi)$  and  $I_2 <_{K^{\text{fin}}}^* \text{EM}_{\{<\}}(I_2, \Phi)$  when we identify  $t \in I_2$  with  $a_t$ .

*Proof.* For  $I_2 \in K^{\text{lin}}$  let the set of elements of  $I_1$  be  $\{\eta : \eta \text{ is a finite sequence of elements from } (\mathbb{Z} \setminus \{0\}) \times I_2\}$ . For  $\eta \in I_1$  let  $(\ell_{\eta, k}, t_{\eta, k})$  be  $\eta(k)$  for  $k < \text{lg}(\eta)$ .

Lastly,  $I_1$  is ordered by:  $\eta_1 < \eta_2$  iff for some  $n$  one of the following occurs

- $\otimes$  (a)  $\eta_1 \upharpoonright n = \eta_2 \upharpoonright n, \text{lg}(\eta_1) > n, \text{lg}(\eta_2) > n$  and  $\ell_{\eta_1, n} < \ell_{\eta_2, n}$
- (b)  $\eta_1 \upharpoonright n = \eta_2 \upharpoonright n, \text{lg}(\eta_1) > n, \text{lg}(\eta_2) > n, \ell_{\eta_1, n} = \ell_{\eta_2, n} > 0$  and  $t_{\eta_1, n} <_{I_2} t_{\eta_2, n}$
- (c)  $\eta_1 \upharpoonright n = \eta_2 \upharpoonright n, \text{lg}(\eta_1) > n, \text{lg}(\eta_2) > n, \ell_{\eta_1, n} = \ell_{\eta_2, n} < 0$  and  $t_{\eta_2, n} <_{I_2} t_{\eta_1, n}$
- (d)  $\eta_1 \upharpoonright n = \eta_2 \upharpoonright n, \text{lg}(\eta_1) = n, \text{lg}(\eta_2) > n$  and  $\ell_{\eta_2, n} > 0$
- (e)  $\eta_1 \upharpoonright n = \eta_2 \upharpoonright n, \text{lg}(\eta_1) > n, \text{lg}(\eta_2) = n$  and  $\ell_{\eta_1, n} < 0$ .

We identify  $t \in I_1$  with the pair  $(1, t)$ . Now check.  $\square_{1.8}$

**1.9 Definition.** 1) Let the language  $\mathbb{L}_{\theta,\partial}[\mathfrak{K}]$  or  $\mathbb{L}_{\theta,\partial,\mathfrak{K}}$  where  $\theta \geq \partial \geq \aleph_0$  and  $\theta$  is possibly  $\infty$ , be defined like the infinitary logic  $\mathbb{L}_{\theta,\partial}(\tau_{\mathfrak{K}})$ , except that we deal only with models from  $K$  and we add for  $i^* < \partial$  the atomic formula “ $\{x_i : i < i^*\}$  is the universe of a  $\leq_{\mathfrak{K}}$ -submodel”, with obvious syntax and semantics. Of course, it is interesting normally only for  $\partial > \text{LS}(\mathfrak{K})$  and recall that any formula has  $< \partial$  free variables.

2) For  $M$  a  $\tau_{\mathfrak{K}}$ -model and  $N \in K$  let  $M \prec_{\mathbb{L}_{\theta,\partial}[\mathfrak{K}]} N$  means that  $M \subseteq N$  and if  $\varphi(\bar{x}, \bar{y})$  is a formula from  $\mathbb{L}_{\theta,\partial}[\mathfrak{K}]$  and  $N \models (\exists \bar{x})\varphi(\bar{x}, \bar{b})$  where  $\bar{b} \in {}^{\ell g(\bar{y})}M$ , then for some  $\bar{a} \in {}^{\ell g(\bar{x})}M$  we have  $N \models \varphi[\bar{a}, \bar{b}]$ .

1.10 Fact: 1) If  $\theta \geq \partial > \text{LS}(\mathfrak{K})$  and  $M, N$  are  $\tau_{\mathfrak{K}}$ -models and  $N \in K$  and  $M \prec_{\mathbb{L}_{\theta,\partial}[\mathfrak{K}]} N$ , then  $M \leq_{\mathfrak{K}} N$  and  $M \in K$ .

2) The relation  $\prec_{\mathbb{L}_{\theta,\partial}[\mathfrak{K}]}$  can also be defined as usual:  $M \prec_{\mathbb{L}_{\theta,\partial}[\mathfrak{K}]} N$  iff  $M, N \in K, M \subseteq N$  and for every  $\varphi(\bar{x}) \in \mathbb{L}_{\theta,\partial}[\mathfrak{K}]$  and  $\bar{a} \in {}^{\ell g(\bar{x})}M$  we have  $M \models \varphi[\bar{a}]$  iff  $N \models \varphi[\bar{a}]$ .

3) If  $N \in \mathfrak{K}$  and  $M$  is a  $\tau_K$ -model satisfying  $M \prec_{\mathbb{L}_{\infty,\kappa}} N$  and  $\kappa > \text{LS}(\mathfrak{K})$  then  $M \in K, M \leq_{\mathfrak{K}} N$  and  $M \prec_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} N$ .

4) If  $N \in K, M$  a  $\tau_K$ -model and  $M \equiv_{\mathbb{L}_{\infty,\kappa}} N$  where  $\kappa > \text{LS}(\mathfrak{K})$  then  $M \in K$  and  $M \equiv_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} N$ .

5) The parallel of 0.17(2) holds for  $\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$ , i.e. there is  $\mathcal{F}$  satisfying clauses (a),(b) there and

(d) if  $f \in \mathcal{F}$  then

( $\alpha$ )  $M \upharpoonright \text{Dom}(f) \leq_{\mathfrak{K}} M$

( $\beta$ )  $N \upharpoonright \text{Rang}(f) \leq_{\mathfrak{K}} M$ .

6) Also the parallel of 0.17(2A) holds for  $\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$ .

7) The parallel of 0.17(4) holds for  $\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$ .

*Proof.* Part (1) is straight (knowing I§1 or [Sh 88, §1]). Part (2) is proved as in the Tarski-Vaught criterion and parts (5),(6),(7) are proved as in 0.17.

Toward proving parts (3),(4) we first assume just

$\boxtimes_1$   $M, N$  are  $\tau_K$ -models,  $N \in K$  and  $M \equiv_{\mathbb{L}_{\infty,\kappa}} N$  and  $\kappa > \text{LS}(\mathfrak{K})$  and  $\lambda \in [\text{LS}(\mathfrak{K}), \kappa)$

and we define:

$\boxdot$  (a)  $I = I_{\lambda} = \{(f, M', N') : M' \subseteq M \text{ and } N' \subseteq N \text{ and } f \text{ is an isomorphism}$

from  $M'$  onto  $N'$  and  $\|M'\| \leq \lambda$  and letting  $\bar{a}$  list  $M'$  we have  
 $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty, \kappa}} (N, f(\bar{a}))$

- (b) for  $t \in I$  let  $t = (f_t, M_t, N_t)$
- (c) for  $\ell = 0, 1, 2$  we define the two-place relation  $\leq_I^\ell$  on  $I$ :  
 let  $s \leq_I^\ell t$  hold iff
  - ( $\alpha$ )  $\ell = 0$  and  $M_s \subseteq M_t \wedge N_s \subseteq N_t$
  - ( $\beta$ )  $\ell = 1$  and  $M_s \leq_{\mathfrak{K}} M_t \wedge N_s \leq_{\mathfrak{K}} N_t$
  - ( $\gamma$ )  $\ell = 2$  and  $f_s \subseteq f_t$

- (d)  $I_1 = I_\lambda^1 := \{t \in I_0 : N_t \leq_{\mathfrak{K}} N\}$  and let  $\leq_{I_1}^\ell = \leq_I^\ell \upharpoonright I_1$  for  $\ell = 0, 1, 2$ .

Now easily

- (\*)<sub>0</sub> ( $\alpha$ )  $I \neq \emptyset$  is partially ordered by  $\leq_I^\ell$  for  $\ell = 0, 1, 2$
- ( $\beta$ )  $s \leq_I^1 t \Rightarrow s \leq_I^0 t$
- ( $\gamma$ )  $s \leq_I^2 t \Rightarrow s \leq_I^0 t$ .

[Why? Straight, e.g.  $I \neq \emptyset$  by 0.17(1).]

- (\*)<sub>1</sub> if  $t \in I_1$  then  $M_t \in K_{\leq \lambda}$  and  $N_t \in K_{\leq \lambda}$ .

[Why? As  $t \in I_1$  by the definition of  $I$  we have  $N_t \in K_{\leq \lambda}$  (because  $N_t \leq_{\mathfrak{K}} N$ ) and  $M_t \in K_{\leq \lambda}$  as  $f_t$  is an isomorphism from  $M_t$  onto  $N_t$ .]

- (\*)<sub>2</sub> if  $s \in I, A \in [M]^{\leq \lambda}$  and  $B \in [N]^{< \lambda}$  then for some  $t$  we have  $s \leq_I^2 t$  and  $A \subseteq M_t$  and  $B \subseteq N_t$ .

[Why? By the properties of  $\equiv_{\mathbb{L}_{\infty, \kappa}}$ , see 0.17(2C) as  $\kappa > \lambda, M \equiv_{\mathbb{L}_{\infty, \kappa}}$  and the definition of  $I$ .]

- (\*)<sub>3</sub> if  $s \leq_{I_1}^2 t$  then  $s \leq_I^1 t$ , i.e.  $M_s \leq_{\mathfrak{K}} M_t$  and  $N_s \leq_{\mathfrak{K}} N_t$ .

[Why? As  $s, t \in I_1$  we know that  $N_s \leq_{\mathfrak{K}} N$  and  $N_t \leq_{\mathfrak{K}} N$  and as  $s \leq_I^2 t$  we have  $f_s \subseteq f_t$  hence  $N_s \subseteq N_t$ . By axiom V of a.e.c. it follows that  $N_s \leq_{\mathfrak{K}} N_t$ . Now  $M_s \leq_{\mathfrak{K}} M_t$  as  $f_t$  is an isomorphism from  $M_t$  onto  $N_t$  mapping  $M_s$  onto  $N_s$  (as it extends  $f_s$  by the definition of  $\leq_I^2$ ) and  $\leq_{\mathfrak{K}}$  is preserved by any isomorphism. So by the definition of  $\leq_I^1$  we are done.]

- (\*)<sub>4</sub> if  $s \in I$  then for some  $t \in I_1$  we have  $s \leq_I^2 t$  (hence  $I_1 \neq \emptyset$ ).

[Why? First choose  $N' \leq_{\mathfrak{K}} N$  of cardinality  $\leq \lambda$  such that  $N_s \subseteq N'$ , (possibly by the basic properties of a.e.c. (see I§1 or Chapter II.B)). Second we can find  $t \in I$  such that  $N_t = N' \wedge f_s \subseteq f_t$  by the characterization of  $\equiv_{\mathbb{L}_{\infty, \kappa}}$  as in  $(*)_2$ . So  $s \leq_I^2 t$  by the definition of  $\leq_I^2$  and  $N_t = N' \leq_{\mathfrak{K}} N$  hence  $t \in I_1$  as required. Lastly,  $I_1 \neq \emptyset$  as by  $(*)_0(\alpha)$  we know that  $I \neq \emptyset$  and apply what we prove.]

$(*)_5$  if  $s \leq_{I_1}^0 t$  then  $N_s \leq_{\mathfrak{K}} N_t$ .

[Why? As in the proof of  $(*)_3$  by AxV of a.e.c. we have  $N_s \leq_{\mathfrak{K}} N_t$  (not the part on the  $M$ 's!)]

$(*)_6$  if  $s \in I_1, A \in [M]^{\leq \lambda}$  and  $B \in [M]^{\leq \lambda}$  then for some  $t$  we have  $s \leq_{I_1}^2 t$  and  $A \subseteq M_t, B \subseteq N_t$ .

[Why? By  $(*)_2$  there is  $t_1$  such that  $s \leq_{I_1}^2 t_1, A \subseteq M_{t_1}$  and  $B \subseteq N_{t_1}$ . By  $(*)_4$  there is  $t \in I_1$  such that  $t_1 \leq_{I_1}^2 t$  hence by  $(*)_0(\alpha)$  we have  $s \leq_{I_1}^2 t$ . As  $s, t \in I_1$  this implies  $s \leq_{I_1}^2 t$ .]

Note that it is unreasonable to have “ $(I_1, \leq_{I_1}^2)$ -directed” but

$(*)_7$   $(I_1, \leq_{I_1}^1)$  is directed.

[Why? Let  $s_1, s_2 \in I_1$ . We now choose  $t_n$  by induction on  $n < \omega$  such that

- (a)  $t_n \in I_1$
- (b)  $M_{t_n}$  includes  $\cup\{M_{t_k} : k < n\} \cup M_{s_1} \cup M_{s_2}$  if  $n \geq 2$
- (c)  $N_{t_n}$  includes  $\cup\{N_{t_k} : k < n\} \cup N_{s_1} \cup N_{s_2}$  if  $n \geq 2$
- (d)  $t_0 = s_1$
- (e)  $t_1 = s_2$
- (f) if  $n = m + 1 \geq 2$  then  $t_m \leq_{I_1}^0 t_n$
- (g) if  $n = m + 2$  then  $t_m \leq_I^2 t_n$  hence  $t_m \leq_{I_1}^2 t_n$ .

For  $n = 0, 1$  this is trivial. For  $n = m + 2 \geq 2$ , apply  $(*)_6$  with  $t_m, \cup\{M_{t_k} : k \leq m + 1\}, \cup\{N_{t_k} : k \leq m + 1\}$  here standing for  $s, A, B$  there getting  $t_n$ , so we get  $t_n \in I_1$  in particular  $t_m \leq_{I_1}^2 t_n$ , so clause (a) is satisfied by  $t_n$ . By the choice of  $t_n$  and as  $s_1 = t_0, s_2 = t_1$ , clauses (b) + (c) hold for  $t_n$ . By the choice of  $t_n$ , obviously also clause (g). Now why does clause (f) hold (i.e.  $t_{m+1} \leq_I^0 t_n$ )? It follows from clauses (a),(b),(c), so  $t_n$  is as required. Hence we have carried the induction. Let  $N^* = \cup\{N_{t_n} : 2 \leq n < \omega\}$ , so clearly by  $(*)_5$  and clause (f) we have  $N_{t_n} \leq_{\mathfrak{K}} N_{t_{n+1}}$  for  $n \geq 1$ , and clearly  $M_{t_n} \subseteq M_{t_{n+1}}$  for  $n \geq 1$ . Let  $M^* = \cup\{M_{t_n} : 2 \leq n < \omega\}$ . Note that by  $(*)_3$  and clause (g) we have  $M_{t_n} \leq_{\mathfrak{K}} M_{t_{n+2}}$ , so  $\langle M_{t_{n+2}} : n < \omega \rangle$  is  $\subseteq$ -increasing, and for  $\ell = 0, 1$  the sequence  $\langle M_{t_{2n+\ell}} : n < \omega \rangle$  is  $\leq_{\mathfrak{K}}$ -increasing



with union  $M^*$ , hence by the basic properties of a.e.c. we have  $M_{2n+\ell} \leq_{\mathfrak{K}} M^*$ . So  $M_{s_1} = M_{t_0} \leq_{\mathfrak{K}} M^*$ ,  $M_{s_2} = M_{t_1} \leq_{\mathfrak{K}} M^*$ . Now  $M_{s_1}, M_{s_2} \subseteq M_{t_2} \leq_{\mathfrak{K}} M^*$  hence  $M_{s_1}, M_{s_2} \leq_{\mathfrak{K}} M_{t_2}$ , Recall that  $N_{s_1} = N_{t_0} \leq_{\mathfrak{K}} N_{t_2}$  was proved above and  $N_{s_2} = N_{t_1} \leq_{\mathfrak{K}} N_{t_2}$  was also proved above so  $t_2$  is a common  $\leq_I^1$ -upper bound of  $s_1, s_2$  as required.]

(\*)<sub>8</sub> if  $s \leq_{I_1}^0 t$  then  $s \leq_{I_1}^1 t$ .

[Why? By (\*)<sub>7</sub> there is  $t_1 \in I_1$  which is a common  $\leq_{I_1}^1$ -upper bound of  $s, t$ . So  $M_s \subseteq M_{t_1}$  (as  $s \leq_{I_1}^0 t$ ) and  $M_s \leq_{\mathfrak{K}} M_{t_1}$  (as  $s \leq_{I_1}^1 t_1$ ) and  $M_t \leq_{\mathfrak{K}} M_{t_1}$  (as  $t \leq_{I_1}^1 t_1$ ). Together by axiom V of a.e.c. we get  $M_s \leq_{\mathfrak{K}} M_t$  and by (\*)<sub>5</sub> we have  $N_s \leq_{\mathfrak{K}} N_t$ . Together  $s \leq_{I_1}^1 t$  as required.]

(\*)<sub>9</sub>  $\langle M_s : s \in (I_1, \leq_{I_1}^1) \rangle$  is  $\leq_{\mathfrak{K}}$ -increasing,  $(I_1, \leq_{I_1}^1)$  is directed and  $\cup \{M_s : s \in I_1\} = M$ .

[Why? The first phrase by the definition of  $\leq_{I_1}^1$  in clause (c)( $\beta$ ) of  $\Box$ , the second by (\*)<sub>7</sub> and the third by (\*)<sub>6</sub> + (\*)<sub>4</sub>.]

By the basic properties of a.e.c. (see I.  88r-1.6 ) we deduce

- ⊙ (a)  $M \in K$
- (b)  $t \in I_1 \Rightarrow M_t \leq_{\mathfrak{K}} M$ .

Now we strengthen the assumption  $\Box_1$  to

$\Box_2$  the demands in  $\Box_1$  and  $M \prec_{\mathbb{L}_{\infty, \kappa}[\tau_{\mathfrak{K}}]} N$ .



We note

- ⊗<sub>1</sub> (a) if  $\bar{a} \in {}^\alpha M$ ,  $|\alpha| + \text{LS}(\mathfrak{K}) \leq \lambda < \kappa$  then for some  $t \in I_\lambda$ ,  $f_t(\bar{a}) = \bar{a}$
- (b) if  $M' \subseteq M$  and  $\|M'\| \leq \lambda$  then  $(\text{id}_{M'}, M', M') \in I_\lambda$
- (c) if  $M_1 \subseteq N_1 \subseteq N$  and  $M_1 \subseteq M$  and  $\|N_1\| \leq \lambda$  then for some  $t \in I$  we have  $N_t = N_1$  and  $\text{id}_{M_1} \subseteq f_t$ .

[Why? Clause (a) is a special case of clause (b) and clause (b) is a special case of clause (c). Lastly, clause (c) follows from the assumption  $M \prec_{\mathbb{L}_{\infty, \kappa}[\tau_{\mathfrak{K}}]} N$  and 0.17(2A),(2B).]

We next shall prove

⊗<sub>2</sub>  $M \leq_{\mathfrak{K}} N$ .

By I.  88r-1.6  and (\*)<sub>9</sub> above for proving ⊗<sub>2</sub> it suffices to prove:

⊗<sub>3</sub> if  $s \in I_1$  then  $M_s \leq_{\mathfrak{K}} N$ .

[Why  $\otimes_3$  holds? As  $M \subseteq N$  there is  $N_* \leq_{\mathfrak{K}} N$  of cardinality  $\leq \lambda$  such that  $M_s \cup N_s \subseteq N_*$ . By  $\otimes_1(c)$  there is  $t \in I$  such that  $N_t = N_*$  and  $\text{id}_{M_s} \subseteq f_t$ . As  $N_* \leq_{\mathfrak{K}} N$  it follows that  $t \in I_1$ . So by  $\boxtimes_1 \Rightarrow \odot(b)$  applied to  $s$  and to  $t$  we can deduce  $M_s \leq_{\mathfrak{K}} M$  and  $M_t \leq_{\mathfrak{K}} M$ . But as  $\text{id}_{M_s} \subseteq f_t$  it follows that  $M_s \subseteq M_t$  hence by AxV of a.e.c. we know that  $M_s \leq_{\mathfrak{K}} M_t$ . But as  $t \in I$  clearly  $f_t$  is an isomorphism from  $N_t$  onto  $M_t$  hence  $f_t^{-1}(M_s) \leq_{\mathfrak{K}} N_t$ , and as  $\text{id}_{M_s} \subseteq f_t$  this means that  $M_s = f_t^{-1}(M_s) \leq_{\mathfrak{K}} N_t$ . Recalling  $N_t \leq_{\mathfrak{K}} N$  and  $\leq_{\mathfrak{K}}$  is transitive it follows that  $M_s \leq_{\mathfrak{K}} N$  as required.]

Let us check parts (3) and (4) of the Fact. Having proved  $\boxtimes_1 \Rightarrow \odot(a)$ , clearly in part (4) of the fact the first conclusion there,  $M \in K$ , holds. The second conclusion,  $M \equiv_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]} N$  holds by

$$\otimes_4 \text{ if } \varphi(\bar{x}) \in \mathbb{L}_{\infty, \kappa}[\mathfrak{K}] \text{ and } |\ell g(\bar{x})| + \text{LS}(\mathfrak{K}) \leq \lambda < \kappa \text{ and } t \in I \text{ and } \bar{a} \in {}^{\ell g(\bar{x})}(M_t) \\ \text{then } M \models \varphi[\bar{a}] \Leftrightarrow N \models \varphi[f_t(\bar{a})].$$

[Why? Prove by induction on the depth of  $\varphi$  for all  $\lambda$  simultaneously. For  $\alpha = 0$ , first for the usual atomic formulas this should be clear. Second, by  $(*)_4$  there is  $t_1$  such that  $t \leq_I^2 t_1 \in I_1$  hence by  $\otimes_3 +$  clause (d) of  $\boxplus +$  clause (b) of  $\odot$  we have  $M_{t_1} \leq_{\mathfrak{K}} N \wedge N_{t_1} \leq_{\mathfrak{K}} N \wedge M_{t_1} \leq_{\mathfrak{K}} M$  respectively. So if  $u \subseteq \ell g(\bar{x})$  then  $M \restriction \text{Rang}(\bar{a} \restriction u) \leq_{\mathfrak{K}} M \Leftrightarrow M \restriction \text{Rang}(\bar{a} \restriction u) \leq_{\mathfrak{K}} M_{t_1} \Leftrightarrow N \restriction \text{Rang}(f(\bar{a}) \restriction u) \leq_{\mathfrak{K}} N_{t_1} \Leftrightarrow N \restriction \text{Rang}(f(\bar{a}) \restriction u) \leq_{\mathfrak{K}} N$ . So we have finished the case of atomic formulas, i.e.  $\alpha = 0$ . For  $\varphi(\bar{x}) = (\exists \bar{y})\psi(\bar{x}, \bar{y})$  use  $(*)_2$ , the other cases are obvious.]

So part (4) holds. As for part (3), the first statement, “ $M \in K$ ” holds by part (4), the second statement,  $M \leq_{\mathfrak{K}} N$ , holds by  $\otimes_2$  and the third statement,  $M \prec_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]} N$  follows by  $\otimes_1(b) + \otimes_4$ . As we have already noted parts (1),(2),(5),(6) and part (7) is proved as  $\otimes_4$  is proved, we are done.  $\square_{1.10}$

**1.11 Claim.** *For a limit cardinal  $\kappa > \text{LS}(\mathfrak{K})$ :*

1)  $M \prec_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]} N$  provided that

- (a) if  $\theta < \kappa$  and  $\theta \in (\text{LS}(\mathfrak{K}), \kappa)$  then  $M \prec_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]} N$
- (b) for every  $\partial < \kappa$  for some  $\theta \in (\partial, \kappa)$  we have: if  $\bar{a}, \bar{b} \in {}^\partial M$  and  $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]} (M, \bar{b})$  then  $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty, \theta_1}[\mathfrak{K}]} (M, \bar{b})$  for every  $\theta_1 \in [\theta, \kappa)$ .

1A)  $M \equiv_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]} N$  provided that

- (a) if  $\text{LS}(\mathfrak{K}) < \theta < \kappa$  then  $M \equiv_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]} N$
- (b) as in part (1).

2) In parts (1) and (1A) we can conclude

- (b)<sup>+</sup> for every  $\partial < \kappa$  for some  $\theta \in (\partial, \kappa)$  we have: if  $\bar{a}, \bar{b} \in {}^\partial M$  and  $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]} (M, \bar{b})$  then  $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]} (M, \bar{b})$ .

3) If  $\text{cf}(\kappa) = \aleph_0$  then  $M \cong N$  when

- (a) if  $\theta < \kappa$  and  $\theta \in (\text{LS}(\mathfrak{K}), \kappa)$  then  $M \equiv_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]} N$
- (b) as in part (1), i.e., for every  $\partial \in (\text{LS}(\mathfrak{K}), \kappa)$  for some  $\theta \in (\partial, \kappa)$  we have: if  $\bar{a} \in {}^\partial M$  and  $\bar{b} \in {}^\partial N$  and  $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]} (N, \bar{b})$  then  $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty, \theta_1}[\mathfrak{K}]} (N, \bar{b})$  for every  $\theta_1 \in (\theta, \kappa)$
- (c)  $M, N$  have cardinality  $\kappa$ .

*Proof.* 1) By 1.10(3) it suffices to prove  $M \prec_{\mathbb{L}_{\infty, \kappa}} N$ , for this it suffices to apply the criterion from 0.17(2A).

Let  $\mathcal{F}$  be the set of functions  $f$  such that:

- ⊙ (α)  $\text{Dom}(f) \subseteq M$  has cardinality  $< \kappa$
- (β)  $\text{Rang}(f) \subseteq N$
- (γ) if  $\bar{a}$  lists  $\text{Dom}(f)$  then for every  $\theta \in (\ell g(\bar{a}), \kappa)$  we have  $\text{tp}_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]}(\bar{a}, \emptyset, M) = \text{tp}_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]}(f(\bar{a}), \emptyset, N)$ .

1A) Similarly.

2) Similarly to part (1) using 1.10(4) and 0.17(2) instead 1.10(3), 0.17(2A).

3) Recall 0.17(1). □<sub>1.11</sub>

**1.12 Claim.** 1) Assume 1.3(a) + (b), i.e.  $\mathfrak{K}$  is categorical in  $\mu > \text{LS}(\mathfrak{K})$ . If  $\mu = \mu^{<\kappa}$  and  $\kappa > \text{LS}(\mathfrak{K})$  then for every  $M \leq_{\mathfrak{K}} N$  from  $K_\mu$  we have  $M \prec_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]} N$  (and there are such  $M <_{\mathfrak{K}_\mu} N$ ).

2) Assume  $\mathfrak{K}$  is weakly or just pseudo  $\mu$ -solvable as witnessed by  $\Phi$  (see Definition 1.4 and Claim 1.5) and  $M^* = \text{EM}_{\tau(\mathfrak{K})}(\mu, \Phi)$ . If  $\mu = \mu^{<\kappa}$  and  $\kappa > |\tau_\Phi|$  and  $M \leq_{\mathfrak{K}} N$  are both isomorphic to  $M^*$  then  $M \prec_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]} N$ .

*Proof.* 1) We prove by induction on  $\gamma$  that for any formula  $\varphi(\bar{x})$  from  $\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]$  of quantifier depth  $\leq \gamma$  (and necessarily  $\ell g(\bar{x}) < \kappa$ ) we have

$$(*) \text{ if } M \leq_{\mathfrak{K}} N \text{ are from } K_\mu \text{ and } \bar{a} \in {}^{\ell g(\bar{x})} M \text{ then } M \models \varphi[\bar{a}] \Leftrightarrow N \models \varphi[\bar{a}].$$

If  $\varphi(\bar{x})$  is atomic this is clear (for the “ $\{x_i : i < i^*\}$  is the universe of a  $\leq_{\mathfrak{K}}$ -submodel”, the implication  $\Rightarrow$  holds as  $\leq_{\mathfrak{K}}$  is transitive and the implication  $\Leftarrow$  as  $\mathfrak{K}$  satisfies AxV of a.e.c.).

If  $\varphi(\bar{x})$  is a Boolean combination of formulas for which the assertion was proved, clearly it holds for  $\varphi(\bar{x})$ . So we are left with the case  $\varphi(\bar{x}) = (\exists \bar{y})\psi(\bar{y}, \bar{x})$ , so  $\ell g(\bar{y}) < \kappa$ . The implication  $\Rightarrow$  is trivial by the induction hypothesis and so suppose

that the other fails, say  $N \models \psi[\bar{b}, \bar{a}]$  and  $M \models \neg(\exists \bar{y})\psi(\bar{y}, \bar{a})$ . We choose by induction on  $i < \mu^+$  a model  $M_i \in K_{\mu, \leq \aleph}$ -increasing continuous, and for each  $i$  in addition we choose an isomorphism  $f_i$  from  $M$  onto  $M_i$  and if  $i = j + 1$  we shall choose an isomorphism  $g_j$  from  $N$  onto  $M_{j+1}$  extending  $f_j$ . For  $i = 0$ , let  $M_0 = M$ , for  $i$  limit let  $M_i = \bigcup_{j < i} M_j$ . For any  $i$ , if  $M_i$  was chosen,  $f_i$  exists as  $\aleph$  is categorical in  $\mu$ . Now if  $i = j + 1$  then  $M_j, f_j$  are well defined and clearly we can choose  $M_i = M_{j+1}, g_j$  as required.

By Fodor lemma, as  $\mu = \mu^{<\kappa}$  and the set  $\{\delta < \mu^+ : \text{cf}(\delta) \geq \kappa\}$  is stationary, clearly for some  $\alpha < \beta < \mu^+$  we have  $f_\alpha(\bar{a}) = f_\beta(\bar{a})$ , now (by the choice of  $g_\alpha$ ) we have  $M_{\alpha+1} \models \psi[g_\alpha(\bar{b}), g_\alpha(\bar{a})]$ , hence by the induction hypothesis applied to the pair  $(M_{\alpha+1}, M_\beta)$  we have  $M_\beta \models \psi[g_\alpha(\bar{b}), g_\alpha(\bar{a})]$  so  $M_\beta \models \varphi[g_\alpha(\bar{a})]$ . But  $g_\alpha(\bar{a}) = f_\alpha(\bar{a}) = f_\beta(\bar{a})$ , contradiction to  $M \models \neg\varphi[\bar{a}]$ .

2) The same proof but we restrict ourselves to models in  $K_{[M^*]}$  so, e.g. in (\*) we have  $M, N \in K_{[M^*]}$  recalling that  $\aleph_{[M^*]}$  is a  $\mu$ -a.e.c., see Definition 0.5(3A) and Claim 0.6(7).  $\square_{1.12}$

**1.13 Exercise:** 1) For the proof (of 1.12(1)) it suffices to assume “ $S \subseteq \{\delta < \mu^+ : \text{cf}(\delta) \geq \kappa\}$  is a stationary subset of  $\mu^+$  and  $M^* \in K_\mu$  is locally  $S$ -weakly limit (see I. 88r-3.1 (5)).

2) Similarly we can weaken the demands “ $M^* = \text{EM}_{\tau(\aleph)}(\mu, \Phi)$  and  $(K, \Phi)$  is pseudo solvable” to: for every  $M \leq_{\aleph} N$  isomorphic to  $M^*$  (which  $\in K_\mu$ ) there is a  $\leq_{\aleph}$ -increasing sequence  $\langle M_\alpha : \alpha < \mu^+ \rangle$  such that  $\{\delta < \mu^+ : \text{cf}(\delta) \geq \kappa \text{ and } (M_\delta, M_{\delta+1}) \text{ is isomorphic to } (M, N) \text{ and } M_\delta = \cup\{M_\alpha : \alpha < \delta\}\}$  is a stationary subset of  $\mu^+$ .

**1.14 Claim.** Assume  $\Phi \in \Upsilon_{<\kappa}^{\text{or}}[\aleph]$  satisfies the conclusion of 1.12(2) for  $(\mu, \kappa)$  and  $\text{LS}(\aleph) < \kappa \leq \mu$  and  $J, I_1, I_2$  are linear orders and  $I_1, I_2$  are  $\kappa$ -wide, see Definition 0.14(1). Then

- (a) If  $I_1 \subseteq I_2$  then  $\text{EM}_{\tau(\aleph)}(I_1, \Phi) \prec_{\mathbb{L}_{\infty, \kappa}[\aleph]} \text{EM}_{\tau(\aleph)}(I_2, \Phi)$
- (b) Assume  $J \subseteq I_1, J \subseteq I_2$ ; if  $\varphi(\bar{x}) \in \mathbb{L}_{\infty, \kappa}[\aleph]$  so  $\ell g(\bar{x}) < \kappa$  and  $\bar{a} \in {}^{\ell g(\bar{x})}(\text{EM}(J, \Phi))$ , then  $\text{EM}_{\tau(\aleph)}(I_1, \Phi) \models \varphi[\bar{a}] \Leftrightarrow \text{EM}_{\tau(\aleph)}(I_2, \Phi) \models \varphi[\bar{a}]$
- (c) Assume  $\bar{\sigma} = \langle \sigma_i(\dots, x_{\alpha(i, \ell)}, \dots)_{\ell < \ell(i)} : i < i(*) \rangle$  where  $i(*) < \kappa$ , each  $\sigma_i$  is a  $\tau(\Phi)$ -term,  $\alpha(i, \ell) < \alpha(*) < \kappa$ . If  $\bar{t}^\ell = \langle t_\alpha^\ell : \alpha < \alpha(*) \rangle$  is a sequence of members of  $I_\ell$  for  $\ell = 1, 2$  and  $\bar{t}^1, \bar{t}^2$  realizes the same quantifier free type in  $I_1, I_2$  respectively and  $\bar{a}^\ell = \langle \sigma_i(\dots, a_{t_{\alpha(i, j)}^\ell}, \dots)_{j < j(i)} : i < i(*) \rangle$  for  $\ell = 1, 2$  then  $\bar{a}^1, \bar{a}^2$  realize the same  $\mathbb{L}_{\infty, \kappa}[\aleph]$ -type in  $\text{EM}_{\tau(\aleph)}(I_1, \Phi), \text{EM}_{\tau(\aleph)}(I_2, \Phi)$  respectively.

*Proof of 1.14.*

Clause (a): We prove that for  $\varphi(\bar{x}) \in \mathbb{L}_{\infty, \kappa}[\mathfrak{K}]$  we have

$(*)_{\varphi(\bar{x})}$  if  $I_1 \subseteq I_2$  are  $\kappa$ -wide linear orders of cardinality  $\leq \mu$  and  $\bar{a} \in {}^{\ell g(\bar{x})}(\text{EM}_{\tau(\mathfrak{K})}(I, \Phi))$   
then  $\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \models \varphi[\bar{a}] \Leftrightarrow \text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi) \models \varphi[\bar{a}]$ .

This easily suffices as for any  $I \in K^{\text{lin}}$ , the model  $\text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$  is the direct limit of  $\langle \text{EM}(I', \Phi) : I' \subseteq I \text{ has cardinality } \leq \mu \rangle$ , which is  $\leq_{\mathfrak{K}}$ -increasing and  $\mu^+$ -directed and as we have:

- ⊙  $M^1 \prec_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]} M^2$  when:
  - (a)  $I$  is a  $\kappa$ -directed partial order
  - (b)  $\bar{M} = \langle M_t : t \in I \rangle$
  - (c)  $s <_I t \rightarrow M_s \prec_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]} M_t$
  - (d)  $M^2 = \cup \{M_t : t \in I\}$
  - (e)  $M^1 \in \{M_t : t \in I\}$  or for some  $\kappa$ -directed  $I' \subseteq I$  we have  $M^1 = \cup \{M_t : t \in I'\}$ .

We prove  $(*)_{\varphi(\bar{x})}$  by induction on  $\varphi$  (as in the proof of 1.12 above). The only non-obvious case is  $\varphi(\bar{x}) = (\exists \bar{y})\psi(\bar{y}, \bar{x})$ , so let  $I_1 \subseteq I_2$  be  $\kappa$ -wide linear orders of cardinality  $\leq \mu$  and  $\bar{a} \in {}^{\ell g(\bar{x})}(\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi))$ . Now if  $\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \models \varphi[\bar{a}]$  then for some  $\bar{b} \in {}^{\ell g(\bar{y})}(\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi))$  we have  $\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \models \psi[\bar{b}, \bar{a}]$  hence by the induction hypothesis  $\text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi) \models \psi[\bar{b}, \bar{a}]$  hence by the satisfaction definition  $\text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi) \models \varphi[\bar{a}]$ , so we have proved the implication  $\Rightarrow$ .

For the other implication assume that  $\bar{b} \in {}^{\ell g(\bar{y})}(\text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi))$  and  $\text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi) \models \psi[\bar{b}, \bar{a}]$ . Let  $\theta = |\ell g(\bar{a} \hat{\ } \bar{b})| + \aleph_0$ , so  $\theta < \kappa$  and without loss of generality if  $\kappa$  is singular then  $\theta \geq \text{cf}(\kappa)$ . Hence there is in  $I_1$  a monotonic sequence  $\bar{c} = \langle c_i : i < \theta^+ \rangle$ , without loss of generality it is increasing. Clearly there is  $I^*$  such that  $\bar{a} \hat{\ } \bar{b} \in {}^{\ell g(\bar{x} \hat{\ } \bar{y})}(\text{EM}(I^*, \Phi))$ ,  $I^* \subseteq I_2$ ,  $|I^*| \leq \theta$  and  $\bar{a} \in {}^{\ell g(\bar{x})}(\text{EM}(I^* \cap I_1, \Phi))$  and without loss of generality  $i < \theta^+ \Rightarrow [c_0, c_i]_{I_2} \cap I^* = \emptyset$ .

Similarly without loss of generality

$(*)$   $I_1 \setminus \cup \{[c_0, c_i]_{I_1} : i < \theta^+\}$  is  $\kappa$ -wide or  $\kappa = \theta^+$ .

Let  $J_0 = I_2$ ; we can find  $J_1$  such that  $J_0 = I_2 \subseteq J_1$  and  $J_1 \setminus I_2 = \{d_\alpha : \alpha < \mu \times \theta^+\}$  with  $d_\alpha$  being  $<_{J_1}$ -increasing with  $\alpha$  and  $(\forall x \in I_2)(x <_{J_1} d_\alpha \equiv \bigvee_{i < \theta^+} x <_{J_1} c_i)$ .

As  $\text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi) \models \psi[\bar{b}, \bar{a}]$  and  $I_2 = J_0 \subseteq J_1$ ,  $|J_1| \leq \mu$  and  $I_2$  is  $\kappa$ -wide (and trivially  $J_1$  is  $\kappa$ -wide), by the induction hypothesis  $\text{EM}_{\tau(\mathfrak{K})}(J_1, \Phi) \models \psi[\bar{b}, \bar{a}]$  hence  $\text{EM}_{\tau(\mathfrak{K})}(J_1, \Phi) \models \varphi[\bar{a}]$ . Let  $J_2 = J_1 \upharpoonright \{x : x \in J_1 \setminus J_0 \text{ or } x \in I_1 \setminus \cup \{[c_0, c_i]_{I_1} : i < \theta^+\}\}$ .

$i < \theta^+\}$ . So  $J_1 \supseteq J_2$ , both linear orders have cardinality  $\mu$  and are  $\kappa$ -wide as witnessed by  $\langle d_\alpha : \alpha < \mu \times \theta^+ \rangle$  for both hence the conclusion of 1.12 holds, i.e.  $\text{EM}(J_2, \Phi) \prec_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]} \text{EM}(J_1, \Phi)$ . Also  $I^* \cap I_1 \subseteq J_2$  and recall that  $\bar{a} \in {}^{\ell g(\bar{x})}(\text{EM}(I^* \cap I_1, \Phi))$  hence  $\bar{a} \in {}^{\ell g(\bar{x})}(\text{EM}(J_2, \Phi))$ . However,  $\text{EM}_{\tau(\mathfrak{K})}(J_1, \Phi) \models \varphi[\bar{a}]$ , see above, hence by the last two sentences  $\text{EM}_{\tau(\mathfrak{K})}(J_2, \Phi) \models \varphi[\bar{a}]$ .

So there is  $\bar{b}^* \in {}^{\ell g(\bar{y})}(\text{EM}_{\tau(\mathfrak{K})}(J_2, \Phi))$  such that  $\text{EM}_{\tau(\mathfrak{K})}(J_2, \Phi) \models \psi[\bar{b}^*, \bar{a}]$ . Let  $J^* \subseteq J_2$  be of cardinality  $\theta$  such that  $\bar{b}^* \in {}^{\ell g(\bar{y})}(\text{EM}_{\tau(\mathfrak{K})}(J^*, \Phi))$  and  $I^* \cap I_1 \subseteq J^*$  recalling  $I^* \cap [c_0, c_i]_{I_2} = \emptyset$  for  $i < \theta^+$ . Now let  $u \subseteq \mu \times \theta^+$  be such that  $J^* \setminus I_1 = \{d_\alpha : \alpha \in u\}$  so  $|u| < \theta^+$ . Let  $J_3 = J_2 \upharpoonright \{t : t \in J_2 \cap I_1 \text{ or } t = d_\alpha \wedge \alpha > \sup(u) \text{ or } t = d_\alpha \wedge \alpha \in u\}$ ; as  $\text{cf}(\mu \times \theta^+) = \theta^+ > |u|$ , clearly  $\sup(u) < \mu \times \theta^+$  hence  $|J_3| = \mu$  and  $J_3$  is  $\kappa$ -wide. So by the conclusion of 1.12 (or by the induction hypothesis) also  $\text{EM}_{\tau(\mathfrak{K})}(J_3, \Phi) \models \psi[\bar{b}^*, \bar{a}]$ . Let  $w = \{\alpha < \mu \times \theta^+ : \alpha \in u \text{ or } \alpha > \sup(u) \wedge (\alpha - \sup(u) < \theta^+)\}$ , so  $\text{otp}(w) = \theta^+$ .

Let  $J_4 = (J_3 \cap I_1) \cup \{d_\alpha : \alpha \in w\}$ , so  $J_4$  is  $\kappa$ -wide as witnessed by  $I_1 \setminus \cup \{[c_0, c_i] : i < \theta^+\}$  or by  $\{d_\alpha : \alpha \in w\}$  recalling (\*) above and  $J_4 \subseteq J_3$  and  $J^* \subseteq J_4$  hence  $\bar{a}, \bar{b}^* \subseteq {}^{\kappa}(\text{EM}(J_4, \Phi))$  hence by the induction hypothesis  $\text{EM}_{\tau(\mathfrak{K})}(J_4, \Phi) \models \psi[\bar{b}^*, \bar{a}]$ .

Let  $J_5 = J_4 \cup \{c_i : i < \theta^+\} \setminus \{d_\alpha : \alpha \in w\}$  equivalently  $J_5 = (J_3 \cap I_1) \cup \{c_\alpha : \alpha < \theta^+\} = (I_1 \setminus \cup \{[c_0, c_i]_{I_1} : i < \theta^+\}) \cup \{c_i : i < \theta^+\}$  so  $J_5 \subseteq I_1$  and let  $h : J_4 \rightarrow J_5$  be such that  $h(d_\alpha) = c_{\text{otp}(w \cap \alpha)}$  for  $\alpha \in w$  and  $h(t) = t$  for others, i.e. for  $t \in J_3 \cap I_1$ . So  $h$  is an isomorphism from  $J_4$  onto  $J_5$ . Recalling 0.12 let  $\hat{h}$  be the isomorphism from  $\text{EM}(J_4, \Phi)$  onto  $\text{EM}(J_5, \Phi)$  which  $h$  induces, so clearly  $\hat{h}(\bar{a}) = \bar{a}$ . Hence for some  $\bar{b}^{**}$  we have  $\bar{b}^{**} = \hat{h}(\bar{b}^*) \in {}^{\ell g(\bar{y})}(\text{EM}_{\tau(\mathfrak{K})}(J_5, \Phi))$  and  $\text{EM}_{\tau(\mathfrak{K})}(J_5, \Phi) \models \psi[\bar{b}^{**}, \bar{a}]$ . Note that by the choice of  $\langle c_i : i < \theta^+ \rangle$ , (see (\*) above), we know that  $J_5$  is  $\kappa$ -wide. Also  $J_5 \subseteq I_1$  so by the induction hypothesis applied to  $\psi(\bar{y}, \bar{x})$ ,  $J_5, I_1$  we have  $\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \models \psi[\bar{b}^{**}, \bar{a}]$  hence by the definition of satisfaction  $\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \models \varphi[\bar{a}]$ , so we have finished proving the implication  $\Leftarrow$  hence clause (a).

Clause (b): Without loss of generality for some linear order  $I$  we have  $I_1 \subseteq I, I_2 \subseteq I$  and  $\text{EM}(I_\ell, \Phi) \subseteq \text{EM}(I, \Phi)$  for  $\ell = 1, 2$  and use clause (a) twice.

Clause (c): Easy by now, e.g. using a linear order  $I'$  extending  $I_1, I_2$  which has an automorphism  $h$  such that  $h(t_\alpha^1) = t_\alpha^2$  for  $\alpha < \alpha^*$ .  $\square_{1.14}$

**1.15 Definition.** Fixing  $\Phi \in \Upsilon_{\mathfrak{K}}^{\text{or}}$ .

- 1) For  $\theta \geq \text{LS}(\mathfrak{K})$  let  $K_\theta^*$ , [let  $K_\theta^{**}$ ] [let  $K_\theta^{*,*}$ ] be the family of  $M \in K_\theta$  isomorphic to some  $\text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$  where  $I$  is a linear order of cardinality  $\theta$  [which is  $\theta$ -wide][which  $\in K_\theta^{\text{fin}}$ ]. More accurately we should write  $K_{\Phi, \theta}^*, K_{\Phi, \theta}^{**}, K_{\Phi, \theta}^{*,*}$ ; similarly below.
- 2) Let  $K^*$  is the class  $\cup \{K_\theta^* : \theta \text{ a cardinal} \geq \text{LS}(\mathfrak{K})\}$ , similarly  $K^{*,*}, K_{\geq \lambda}^*, K_{\geq \lambda}^{**}$ , etc.

- 3) Let  $\mathfrak{K}^* = \mathfrak{K}_\Phi^* = (K^*, \leq_\mathfrak{K} \restriction K^*)$ .
- 4) Let  $\mathfrak{K}_\lambda^* = K_{\Phi, \lambda}^*$  be  $(K_{\Phi, \lambda}^*, \leq_\mathfrak{K} \restriction K_{\Phi, \lambda}^*)$ .

**1.16 Claim.** 1)  $K_\theta^{**}$  is categorical in  $\theta$  if  $\text{LS}(\mathfrak{K}) < \theta \leq \mu$ ,  $\text{cf}(\theta) = \aleph_0$  and the conclusion of 1.12(2) hence of 1.14 holds for  $\partial = \theta$  (and  $\Phi$ ), e.g.  $\mathfrak{K}$  is pseudo solvable in  $\mu$  as witnessed by  $\Phi$  and  $\mu = \mu^{<\theta}$ .

2)  $K_\theta^{*,*}, K_\theta^{**} \subseteq K_\theta^*$ .

3) If  $\theta$  is strong limit  $> \text{LS}(\mathfrak{K})$  then  $K_\theta^{**} = K_\theta^*$ .

*Proof.* 1) By 1.14 and 0.17(1).

2) Read the definitions.

3) Recall 0.15(2). □<sub>1.16</sub>

**1.17 Remark.** 1) We will be specially interested in 1.16 in the case  $(\mu, \lambda)$  is a  $\mathfrak{K}$ -candidate (see Definition V.1.3) and  $\theta = \lambda$ .

2) Note that  $K_\theta^*$  in general is not a  $\theta$ -a.e.c.

3) If we strengthen 1.18(2) below, replacing  $(\mu, \lambda)$  by  $(\mu, \lambda^+)$  then categoricity of  $K_\lambda^*$  and in fact Claim 1.19(4) follows immediately from (or as in) Claim 1.16(1).

For the rest of this section we assume that the triple  $(\mu, \lambda, \Phi)$  is a pseudo  $\mathfrak{K}$ -candidate (see Definition 1.3) and rather than  $\mu = \mu^\lambda$  we assume just the conclusion of 1.12, that is:

**1.18 Hypothesis.** 1) The pair  $(\mu, \lambda)$  is a pseudo  $\mathfrak{K}$ -candidate and  $\Phi$  witnesses this, so  $|\tau_\Phi| \leq \text{LS}(\mathfrak{K}) < \lambda = \beth_\lambda < \mu$  and  $\Phi \in \Upsilon_\mathfrak{K}^{\text{or}}$  is as in Definition 1.4 so  $I \in K_\mu^{\text{lin}} \Rightarrow \text{EM}_{\tau(\mathfrak{K})}(I, \Phi) \in K_\mu^{\text{pl}}$ .

2) For every  $\kappa \in (\text{LS}(\mathfrak{K}), \lambda)$  the conclusion of 1.12(2) holds hence also of 1.14 (if  $\mu = \mu^{<\lambda}$  this follows from (1) even for  $\kappa = \lambda^+$  as  $\mu^{<\kappa} = \mu^\lambda = \mu$  by cardinal arithmetic).

**1.19 Claim.** 1) If  $M_1 \leq_\mathfrak{K} M_2$  are from  $K_\lambda^*$  or just  $K_{\geq \lambda}^*$  and  $\text{LS}(\mathfrak{K}) < \theta < \lambda$  then  $M_1 \prec_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]} M_2$ ; moreover  $M_1 \prec_{\mathbb{L}_{\infty, \lambda}[\mathfrak{K}]} M_2$ .

2) If  $M_1 \leq_\mathfrak{K} M_2$  are from  $K^*$  and  $\|M_1\| \geq \kappa := \beth_{1,1}(\theta)$  (recall that this is  $\beth_{(2^\theta)^+}$ ) and  $\mu > \theta \geq \text{LS}(\mathfrak{K})$  then  $M_1 \prec_{\mathbb{L}_{\infty, \theta^+}[\mathfrak{K}]} M_2$ .

3) Assume  $\text{LS}(\mathfrak{K}) < \theta < \kappa = \beth_{1,1}(\theta) \leq \chi < \mu$ ,  $\chi_1 = \beth_{1,1}(\chi)$  and  $M \in K_{\geq \chi_1}^*$  and  $\bar{a}, \bar{b} \in {}^\gamma M$  where  $\gamma < \theta^+$  and  $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]} (M, \bar{b})$ , i.e.  $\varphi(\langle x_\beta : \beta < \gamma \rangle) \in \mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}] \Rightarrow M \models \varphi[\bar{a}] \Leftrightarrow M \models \varphi[\bar{b}]$ . Then  $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty, \chi}[\mathfrak{K}]} (M, \bar{b})$ .

4)  $K_\lambda^*$  is categorical in  $\lambda$  provided that  $\text{cf}(\lambda) = \aleph_0$ .

1.20 *Remark.* 1) What is the difference between say 1.19(3) and clause (a) of 1.14? Here there is no connection between the additional  $\tau(\Phi)$ -structures expanding  $M_1, M_2$ .

2) Note that  $\Phi$  has the  $\kappa$ -non-order property (see 1.5(2)(\*)) when  $\kappa \geq \text{LS}(\mathfrak{K}), \kappa^+ < \mu$  using 1.19(4).

3) Concerning 1.19(2), note that if  $\|M_1\| \geq \mu$  it is easy to deduce this from 1.18(2), i.e., 1.12(2). But the whole point in this stage is to deduce something on cardinals  $< \mu$ .

4) Note that the proof of 1.19(2) gives:

⊗ assume  $\text{LS}(\mathfrak{K}) \leq \theta$  and  $\delta(*) = \text{Min}\{(2^\theta)^+, \delta(2^{\text{LS}(\mathfrak{K})} + \theta)\}$  where on the function  $\delta(-)$ , see II.A. 300a-1.2.3, II.A. 300a-1.2, if  $\beth_{\delta(*)} \leq \mu$  then for some  $\alpha(*) < \delta(*)$  we have:

⊙ if  $M_1 \leq_{\mathfrak{K}} M_2$  are from  $K^*$  and  $\|M_1\| \geq \beth_{\alpha(*)}$  then  $M_1 \prec_{\mathbb{L}_{\infty, \theta+}[\mathfrak{K}]} M_2$ .

5) Similarly for 1.19(3) so we can weaken the demand  $M \in K_{\geq \chi_1}^*$

6) We use “ $\lambda$  has countable cofinality, i.e.  $\text{cf}(\lambda) = \aleph_0$ ” in the proof of part (4) of 1.19, but not in the proof of the other parts.

7) Recall that for notational simplicity we assume  $\text{LS}(\mathfrak{K}) \geq |\tau_{\mathfrak{K}}|$  hence  $\theta \geq |\tau_{\Phi}|$ .

8) Note that for 1.19(2),(3) we can omit  $\lambda$  from Hypothesis 1.18.

9) Note that we shall use not only 1.19 but also its proof.

*Proof of 1.19.* 1) The first phrase holds by part (2) noting that  $\kappa < \lambda$  if  $\theta < \lambda$  as  $\theta < \lambda = \beth_\lambda$ . The second phrase holds by 1.11 as its assumption holds by parts (1) and (3).



2) We prove by induction on the ordinal  $\gamma$  that:

(\*) if  $M_1 \leq_{\mathfrak{K}} M_2$  are from  $K_{\geq \kappa}^*$  and the formula  $\varphi(\bar{x}) \in \mathbb{L}_{\infty, \theta+}[\mathfrak{K}]$  has depth  $\leq \gamma$  (so necessarily  $\ell g(\bar{x}) < \theta^+$ ) and  $\bar{a} \in {}^{\ell g(\bar{x})}(M_1)$  then  $M_1 \models \varphi[\bar{a}] \Leftrightarrow M_2 \models \varphi[\bar{a}]$ .



As in 1.12, the non-trivial case is to assume  $\varphi(\bar{x}) = (\exists \bar{y})\psi(\bar{y}, \bar{x})$  where  $\bar{a} \in {}^{\ell g(\bar{x})}(M_1)$  and  $M_2 \models \varphi[\bar{a}]$  and we shall prove  $M_1 \models \varphi[\bar{a}]$ , so necessarily  $\ell g(\bar{x}) + \ell g(\bar{y}) < \theta^+$  and we can choose  $\bar{b} \in {}^{\ell g(\bar{y})}(M_2)$  such that  $M_2 \models \psi[\bar{b}, \bar{a}]$ . For  $\ell = 1, 2$  as  $M_\ell \in K_{\geq \kappa}^*$  there is an isomorphism  $f_\ell$  from  $\text{EM}_{\tau(\mathfrak{K})}(I_\ell, \Phi)$  onto  $M_\ell$  for some linear order  $I_\ell$  of cardinality  $\geq \kappa$ .

So we can find  $J_\ell \subseteq I_\ell$  of cardinality  $\theta$  for  $\ell = 1, 2$  such that  $\bar{a} \subseteq M_1^-$  where  $M_1^- = f_1(\text{EM}_{\tau(\mathfrak{K})}(J_1, \Phi))$ , and  $\bar{a} \hat{\ } \bar{b} \subseteq M_2^-$  where  $M_2^- = f_2(\text{EM}_{\tau(\mathfrak{K})}(J_2, \Phi))$  and without loss of generality  $M_1^- = M_2^- \cap M_1$ . By 1.18(1), i.e. 0.9(1), clause (c)



clearly  $M_\ell^- \leq_{\mathfrak{K}} M_\ell$  and so by AxV of a.e.c. (see Definition III. 600-0.2 ) , we have  $M_1^- \leq_{\mathfrak{K}} M_2^-$ . First assume  $\theta \geq 2^{\text{LS}(\mathfrak{K})}$ ; in fact it is not a real loss to assume this. By renaming without loss of generality there is a transitive set  $B$  (in the set theoretic sense) of cardinality  $\leq \theta$  such that the following objects belong to it:

- $\oplus(a)$   $J_1, J_2$
- (b)  $\Phi$  (i.e.  $\tau_\Phi$  and  $\langle (\text{EM}(n, \Phi), a_\ell)_{\ell < n} : n < \omega \rangle$ )
- (c)  $\mathfrak{K}$ , i.e.,  $\tau_{\mathfrak{K}}$  and  $\{(M, N) : M \leq_{\mathfrak{K}} N \text{ have universe included in } \text{LS}(\mathfrak{K})\}$
- (d)  $\text{EM}(J_\ell, \Phi)$  and  $\langle a_t : t \in J_\ell \rangle$  for  $\ell = 1, 2$ .





Let  $\chi$  be large enough,  $\mathfrak{B} = (\mathcal{H}(\chi), \in, <_\chi^*)$  and  $\mathfrak{B}^+$  be  $\mathfrak{B}$  expanded by the individual constants  $M_\ell^+ = \text{EM}(I_\ell, \Phi), \langle a_t^\ell : t \in I_\ell \rangle$  the skeleton,  $M_\ell, M_\ell^-$  and  $f_\ell$  (all for  $\ell = 1, 2$ ),  $\kappa, B$  and  $x$  for each  $x \in B$ . By the assumption  $\|M_1\| \geq \kappa = \beth_{1,1}(\theta)$ , hence (see here II.A. 300a-1.2 ) there is  $\mathfrak{C}$  such that

- $\odot$  (a)  $\mathfrak{C}$  is a  $\tau(\mathfrak{B}^+)$ -model elementarily equivalent to  $\mathfrak{B}^+$   
(that is, in first order logic)
- (b)  $\mathfrak{C}$  omits the type  $\{x \neq b \ \& \ x \in B : b \in B\}$  but
- (c)  $|\{b : \mathfrak{C} \models "b \in \kappa^{\mathfrak{C}}"\}| = \mu = \|\mathfrak{C}\|$ .

Without loss of generality  $b \in B \Rightarrow b^{\mathfrak{C}} = b$ .

Now

- $\otimes_1$  if  $\mathfrak{C} \models "M \in K"$ , so  $M$  is just a member of the model  $\mathfrak{C}$  then we can define a  $\tau_{\mathfrak{K}}$ -model  $M^{\mathfrak{C}} = M[\mathfrak{C}]$  as follows
  - (a) the set of elements of  $M^{\mathfrak{C}}$  is  $\{a : \mathfrak{C} \models "a \text{ is a member of the model } M"\}$
  - (b) if  $R \in \tau_K$  is an  $n$ -place predicate then  $R^{M[\mathfrak{C}]} = \{\langle a_\ell : \ell < n \rangle : \mathfrak{C} \models "\langle a_\ell : \ell < n \rangle \in R^M"\}$
  - (c) if  $F \in \tau_K$  is an  $n$ -place function symbol,  $F^{M[\mathfrak{C}]}$  is defined similarly.
- $\otimes_2$  (a) if  $\mathfrak{C} \models "I \text{ is a linear order}"$  then we define  $I^{\mathfrak{C}}$  similarly
- (b) similarly if  $\mathfrak{C} \models "M \text{ is a } \tau(\Phi)\text{-model}"$
- $\otimes_3$  if  $\mathfrak{C} \models "I \text{ is a directed partial order, } \bar{M} = \langle M_s : s \in I \rangle \text{ satisfies } M_s \in K \text{ has cardinality } \text{LS}(\mathfrak{K}) \text{ and } s \leq_I t \Rightarrow M_s \leq_{\mathfrak{K}} M_t"$  then also  $\langle M_s^{\mathfrak{C}} : s \in I^{\mathfrak{C}} \rangle$  satisfies this.

By easy absoluteness (for clauses  $(a)_1, (a)_2$  we use I.  88r-1.6 , I.  88r-1.7  and  $\otimes_3$ ):

- $\boxtimes (a)_1$  if  $\mathfrak{C} \models "M \in K"$  then  $M^\mathfrak{C} \in K$
- $(a)_2$  if  $\mathfrak{C} \models "M \leq_{\mathfrak{K}} N"$  then  $M^\mathfrak{C} \leq_{\mathfrak{K}} N^\mathfrak{C}$
- $(b)_1$  if  $\mathfrak{C} \models "I \text{ is a linear order}"$  then  $I^\mathfrak{C} = I[\mathfrak{C}]$  is a linear order
- $(b)_2$  if  $\mathfrak{C} \models "I \subseteq J \text{ as linear orders}"$  then  $I^\mathfrak{C} \subseteq J^\mathfrak{C}$
- $(c)$  similarly for  $\tau_\Phi$ -models
- $(d)_1$  if  $\mathfrak{C} \models "M = \text{EM}(I, \Phi)"$  then there is a canonical isomorphism  $f_I^\mathfrak{C}$  from  $\text{EM}(I^\mathfrak{C}, \Phi)$  onto  $M^\mathfrak{C}$  (hence it is also an isomorphism from  $\text{EM}_{\tau(\mathfrak{K})}(I^\mathfrak{C}, \Phi)$  onto  $M^\mathfrak{C} \upharpoonright \tau(\mathfrak{K})$ )
- $(d)_2$  if  $\mathfrak{C} \models "I \subseteq J \text{ as linear orders}"$  then  $f_J^\mathfrak{C}$  extends  $f_I^\mathfrak{C}$ .

Now clearly  $J_\ell^\mathfrak{C} = J_\ell$  and  $I_\ell^\mathfrak{C}$  is a linear order of cardinality  $\mu$  extending  $J_\ell$  for  $\ell = 1, 2$ . Let  $M_\ell^* = (M_\ell^-)^\mathfrak{C}$  for  $\ell = 1, 2$ .

So recalling clause (c) of  $\odot$  we have:  $M_1^\mathfrak{C}, M_2^\mathfrak{C} \in K_\mu^*, M_1^\mathfrak{C} \leq_{\mathfrak{K}} M_2^\mathfrak{C}, M_\ell^* \leq_{\mathfrak{K}} M_\ell^\mathfrak{C}, M_1^* \leq_{\mathfrak{K}} M_2^*$  and  $f_\ell^{\mathfrak{C}_0}, f_{I_\ell}^\mathfrak{C}$  are isomorphisms from  $\text{EM}_{\tau(\mathfrak{K})}(I_\ell^\mathfrak{C}, \Phi)$  onto  $M_\ell^\mathfrak{C}$ , in fact,  $f_{I_\ell}^\mathfrak{C}$  is the identity on  $\text{EM}_{\tau(\mathfrak{K})}(J_\ell^\mathfrak{C}, \Phi) = \text{EM}_{\tau(\mathfrak{K})}(J_\ell, \Phi)$  and  $f_\ell^\mathfrak{C}$  maps it onto  $M_\ell^*$  for  $\ell = 1, 2$ .

Now  $M_2 \models \psi[\bar{a}, \bar{b}]$ , (why? assumed above) hence  $M_2^\mathfrak{C} \models \psi[\bar{a}, \bar{b}]$  (why? By 1.14, clause (b) or (c) and the situation recalling 1.18(2), of course noting that  $I_2, I_2^\mathfrak{C}$  are of cardinality  $\geq \kappa = \beth_{1,1}(\theta)$  hence are  $\theta^+$ -wide), hence  $M_2^\mathfrak{C} \models \varphi[\bar{a}]$  (by definition of satisfaction), hence  $M_1^\mathfrak{C} \models \varphi[\bar{a}]$  (why? as  $M_1^\mathfrak{C}, M_2^\mathfrak{C} \in K_\mu^*$  hence  $M_1^\mathfrak{C} \prec_{\mathbb{L}_{\infty, \theta^+}[\mathfrak{K}]} M_2^\mathfrak{C}$  by  $\boxtimes$  and 1.18(2) and recalling 1.12(2)) hence  $M_1 \models \varphi[\bar{a}]$  (why? by clause (b) of 1.14 recalling 1.18(2)) as required in 1.19(2).]

So we are done except for a small debt: the case  $\theta < 2^{\text{LS}(\mathfrak{K})}$  and  $f_\ell^\mathfrak{C}$  is an isomorphism from  $\text{EM}_{\tau(\mathfrak{K})}(I_\ell^\mathfrak{C}, \Phi)$ .

In this case choose two sets  $B_1, B_2$  such that  $|B_1| = \theta, |B_2| = 2^{\text{LS}(\mathfrak{K})}, B_1 \subseteq B_2$  and concerning the demands in  $\oplus$  above the objects from (a),(b),(d) and  $\tau_{\mathfrak{K}}$  belong to  $B_1$ , the objects from (c) belong to  $B_2$ .

Again, without loss of generality  $B_1, B_2$  are transitive sets and  $B_1, B_2$  serve as individual constants of  $\mathfrak{B}^+$  as well as each member of  $B_1$ . Now concerning  $\mathfrak{C}$  we demand that it is elementarily equivalent to  $\mathfrak{B}^+$ ; omit  $\{x \in B_1 \wedge x \neq b : b \in B_1\}$  and for some  $\mathfrak{B}_1^+ \prec \mathfrak{B}^+$  of cardinality  $\theta$  we have  $\mathfrak{B}_1^+ \prec \mathfrak{C}$  and  $\{b : \mathfrak{C} \models b \in B_2\} \subseteq \mathfrak{B}^+$ . This influences just the proof of  $\otimes_3$ .

3) Without loss of generality  $M = \text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$  and  $I \in K_{\geq \chi_1}^{\text{lin}}$ . As  $\gamma < \theta^+$  and  $\bar{a}, \bar{b} \in {}^\gamma M$  there is  $I_1 \subseteq I$  of cardinality  $\theta$  such that  $\bar{a}, \bar{b} \in {}^\gamma(M_1)$  where  $M_1 = \text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi)$ . As  $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]} (M, \bar{b})$  necessarily there is  $I_2 \subseteq I$  of cardinality  $\kappa$  and automorphism  $f$  of  $M_2 = \text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$  mapping  $\bar{a}$  to  $\bar{b}$  such

that  $I_1 \subseteq I_2$ . Why? Recalling 0.17(2), by the hence and forth argument as in the second part of the proof of 1.10(3).

Now as in the proof of part (2) there is a linear order  $I_3$  extending  $I_1$  of cardinality  $\chi_1$  and an automorphism  $g$  of  $M_3 = \text{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$  mapping  $\bar{a}$  to  $\bar{b}$ . Without loss of generality for some linear order  $I_4$  we have  $I \subseteq I_4$  and  $I_3 \subseteq I_4$ .

Let  $M_4 = \text{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$ , now  $M \prec_{\mathbb{L}_{\infty, \chi^+}[\mathfrak{K}]} M_4$  by part (2),  $M_3 \prec_{\mathbb{L}_{\infty, \chi^+}[\mathfrak{K}]} M_4$  by part (3) and  $(M_3, \bar{a}) \equiv_{\mathbb{L}_{\infty, \chi^+}[\mathfrak{K}]} (M_3, \bar{b})$  by using the automorphism  $g$  of  $M_3$  so together we are done.

4) So let  $M, N \in K_\lambda^*$  (in fact, hence  $\in K_\lambda^{**}$  recalling  $K_\lambda^* = K_\lambda^{**}$  by 1.16(3) but not used). By parts (1),(3) the assumptions of 1.11(3) holds with  $\lambda$  here standing for  $\kappa$  there, hence its conclusion, i.e.  $M \cong N$ .

□<sub>1.19</sub>

Note: here the types below are sets of formulas.

**1.21 Definition.** Assume  $M \in K, \mathbf{I} \subseteq {}^\gamma M$  and  $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$  are languages in the vocabulary  $\tau_{\mathfrak{K}}$ .

1) We say that  $\mathbf{I}$  is  $(\mathcal{L}, \partial, < \kappa)$ -convergent in  $M$ , if:  $|\mathbf{I}| \geq \partial$  and for every  $\bar{b} \in {}^{\kappa}M$ , for some  $\mathbf{J} \subseteq \mathbf{I}$  of cardinality  $< \partial$  for some<sup>2</sup>  $p$  we have:

(\*) for every  $\bar{c} \in \mathbf{I} \setminus \mathbf{J}$ , the  $\mathcal{L}$ -type of  $\bar{c} \hat{\ } \bar{b}$  in  $M$  is  $p$ .

2) Let  $\text{Av}_{\mathcal{L}, \partial, < \kappa}(\mathbf{I}, M) = \{\varphi(\bar{x}, \bar{b}) : \varphi(\bar{x}, \bar{y}) \text{ is an } \mathcal{L}\text{-formula, } \ell g(\bar{y}) < \kappa \text{ and } \bar{a} \in \mathbf{I} \Rightarrow \ell g(\bar{a}) = \ell g(\bar{x}) \text{ and } \bar{b} \in {}^{\ell g(\bar{y})}M \text{ and for all but } < \partial \text{ of the sequences } \bar{c} \in \mathbf{I}, \text{ the sequence } \bar{c} \text{ satisfies } \varphi(\bar{x}, \bar{b}) \text{ in } M\}$ . If  $\partial$  is missing, we mean  $\partial = \kappa$ . In parts (1) and (2) we may write “ $\kappa$ ” instead of  $< \kappa^+$ ; similarly below.

3) We say that  $\mathbf{I}$  is  $(\mathcal{L}_1, \mathcal{L}_2, \partial, < \kappa)$ -based on  $A$  in  $M$  (if  $\mathcal{L}_1 = \mathcal{L} = \mathcal{L}_2$  we may write only  $\mathcal{L}$ ) when:

(a)  $A \subseteq M$

(b)  $\mathbf{I}$  is  $(\mathcal{L}_1, \partial, < \kappa)$ -convergent,

(c)  $\text{Av}_{\mathcal{L}_1, \partial, < \kappa}(\mathbf{I}, M)$  does not  $(\mathcal{L}_1, \mathcal{L}_2, < \kappa)$ -split over  $A$ , see below.

4) We say that  $p(\bar{x}) \in \text{Sfr}_{\mathcal{L}}^\alpha(B, M)$  does not  $(\mathcal{L}_1, \mathcal{L}_2, < \kappa)$ -split over  $A$  when: if  $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_1, \alpha = \ell g(\bar{x}) < \kappa, \ell g(\bar{y}) < \kappa$  and  $\bar{b}, \bar{c} \in {}^{\ell g(\bar{y})}B$  realize the same  $\mathcal{L}_2$ -type in  $M$  over  $A$  then  $\varphi(\bar{x}, \bar{b}) \in p \Leftrightarrow \varphi(\bar{x}, \bar{c}) \in p$ ; recalling that  $\text{Sfr}_{\mathcal{L}}^\alpha(A, M)$  is defined in 0.4 and normally  $\mathcal{L}_1 = \mathcal{L}_2$  or at least  $\mathcal{L}_1 \subseteq \mathcal{L}_2$ .

5) Let  $\text{Av}_{< \kappa}(\mathbf{I}, M)$  be  $\text{Av}_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]}(\mathbf{I}, M)$  and  $\text{Av}_\kappa(\mathbf{I}, M)$  be  $\text{Av}_{\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]}(\mathbf{I}, M)$ .

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<sup>2</sup>We could have demanded it for every single formula, here this distinction is not important

1.22 Remark. 1) See definition of  $\text{Sav}^\alpha(M)$  in 1.34(2) below.

2) An alternative for clause (c) of 1.21(3) is:

(c)' the set  $\{\text{Av}_{\mathcal{L}, \partial, < \kappa}(f(\mathbf{I}), M) : f \text{ an automorphism of } M \text{ over } A\}$  has cardinality  $\leq \beth_{1,1}(\text{LS}(\mathfrak{K}) + \theta + |A|) < \|M\|$ .

**1.23 Claim.** 1) Assume that  $M \in K, A \subseteq M, \mathbf{I} \subseteq {}^\theta M, |\mathbf{I}| \geq \partial = \text{cf}(\partial) > \kappa \geq \theta + \text{LS}(\mathfrak{K})$  and  $\mathbf{I}$  is  $(\mathcal{L}, \partial, \kappa)$ -convergent. Then the type  $p = \text{Av}_{\mathcal{L}, \partial, \kappa}(\mathbf{I}, M)$  belongs to  $\text{Sfr}_{\mathcal{L}}^\theta(M)$ , i.e., it is complete, recalling Definition 0.4 (no demand that it is realized in some  $N, M \leq_{\mathfrak{K}} N!$ ).

2) Also  $\mathbf{I}$  is  $(\mathcal{L}, \partial, \kappa)$ -based on some set of cardinality  $\leq \partial$ , even on  $\cup \mathbf{J}$ , for any  $\mathbf{J} \subseteq \mathbf{I}$  of cardinality  $\geq \partial$ .

*Proof.* 1) By the definition.

2) By the definitions: if  $\bar{b} \in {}^{\kappa^+}M, \varphi = \varphi(\bar{x}, \bar{y}) \in \mathcal{L}$  and  $\ell g(\bar{b}) = \ell g(\bar{y}), \ell g(\bar{x}) = \theta$ , then by the convergence

$$\begin{aligned} \varphi(\bar{x}, \bar{b}) \in p &\Leftrightarrow \text{for all but } < \partial \text{ members } \bar{a} \text{ of } \mathbf{I}, M \models \varphi[\bar{a}, \bar{b}] \Leftrightarrow \\ &\text{for all but } < \partial \text{ members of } \mathbf{J}, M \models \varphi[\bar{a}, \bar{b}]. \end{aligned}$$

So only  $\text{tp}_{\mathcal{L}}(\bar{b}, \cup \mathbf{J}, M)$  matters hence the non-splitting required in clause (c) of Definition 1.21(3).  $\square_{1.23}$

As in II.A. 300a-1.7, we deduce non-splitting over a small set from non-order.

**1.24 Claim.** Assume  $M = \text{EM}_{\tau(\mathfrak{K})}(I, \Phi), \theta + \text{LS}(\mathfrak{K}) \leq \kappa < \lambda$  and  $\beth_{1,1}(\partial) \leq |I|$  where  $\partial = (2^{2^\kappa})^+$  or  $I$  is well ordered and  $\partial = (2^\kappa)^+$ . If  $M \prec_{\mathbb{L}_{\infty, \partial}[\mathfrak{K}]} N$  then for every  $\bar{a} \in {}^{\theta \geq N}$  there is  $B \subseteq M$  of cardinality  $< \partial$  such that  $\text{tp}_{\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]}(\bar{a}, M, N)$  does not  $(\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}], \mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}])$ -split over  $B$ .

*Proof.* Let  $\bar{x} = \langle x_i : i < \ell g(\bar{a}) \rangle$ .

We try to choose  $B_\alpha, \gamma_\alpha, \bar{a}_\alpha, \bar{b}_\alpha, \bar{c}_\alpha, \varphi_\alpha(\bar{x}, \bar{y}_\alpha) \in \mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]$  by induction on  $\alpha < \partial$  such that

- ⊗ (a)  $B_\alpha = \cup \{\bar{a}_\beta : \beta < \alpha\}$
- (b)  $\bar{b}_\alpha, \bar{c}_\alpha \in {}^{\gamma_\alpha}M$  and  $\gamma_\alpha < \kappa^+$

- (c)  $\varphi_\alpha(\bar{x}, \bar{y}_\alpha) \in \mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]$  such that  $\ell g(\bar{y}_\alpha) = \gamma_\alpha$
- (d)  $N \models \text{"}\varphi_\alpha[\bar{a}, \bar{b}_\alpha] \equiv \neg \varphi_\alpha[\bar{a}, \bar{c}_\alpha]\text{"}$
- (e)  $\bar{a}_\alpha \in {}^{\ell g(\bar{a})}M$  realizes  $\{\varphi_\beta(\bar{x}, \bar{b}_\beta) \equiv \neg \varphi_\beta(\bar{x}, \bar{c}_\beta) : \beta < \alpha\}$  in  $M$
- (f)  $M \models \text{"}\varphi_\alpha[\bar{a}_\beta, \bar{b}_\alpha] \equiv \varphi_\alpha[\bar{a}_\beta, \bar{c}_\alpha]\text{"}$  for  $\beta \leq \alpha$ .

If we are stuck at  $\alpha(*) < \partial$  then we cannot choose  $\gamma_\alpha, \bar{b}_\alpha, \bar{c}_\alpha, \varphi_\alpha(\bar{x}, \bar{y}_\alpha)$  clauses (b),(c),(d), because then  $\bar{a}_\alpha$  as required in clauses (e),(f) exists because  $M \prec_{\mathbb{L}_{\infty, \partial}[\mathfrak{K}]} N$ . Hence  $B := \cup\{\bar{a}_\alpha : \alpha < \alpha(*)\}$  is as required. So assume that we have carried the induction. As  $\gamma_\alpha < \kappa^+ < \partial = \text{cf}(\partial)$  without loss of generality  $\gamma_\alpha = \gamma < \kappa^+$  for every  $\alpha < \partial$ .

Let  $\partial_1 = (2^\kappa)^+$ .

Now by 1.25(5) below when  $I$  is not well ordered and by 1.25(4) below when  $I$  is well ordered (and part (1) of 1.25(1), recalling  $I$  is  $\kappa^+$ -wide as  $\kappa < \partial$  and  $\beth_{1,1}(\partial) \leq |I|$ ) clearly for some  $S \subseteq \partial$  of order type  $\partial_1$ , the sequence  $\langle \bar{a}_\alpha \hat{\ } \bar{b}_\alpha \hat{\ } \bar{c}_\alpha : \alpha \in S \rangle$  is  $(\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}], \kappa^+, \kappa)$ -convergent and  $(\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}], < \omega)$ -indiscernible in  $M$  hence without loss of generality  $\alpha \in S \Rightarrow \varphi_\alpha = \varphi$ . But as  $\partial_1 > \kappa^+$  this contradicts (e) + (f) of  $\otimes$  (if we use  $\partial_1 = \kappa^+$ , we can use a further conclusion of 1.25(1) stated in 1.25(2), i.e.,  $\langle \bar{a}_\alpha \hat{\ } \bar{b}_\alpha \hat{\ } \bar{c}_\alpha : \alpha \in S \rangle$  is a  $(\mathbb{L}_{\infty, \kappa}[\mathfrak{K}], < \omega)$ -indiscernible set not just a sequence, contradiction to (e) + (f) of  $\otimes$ ).  $\square_{1.24}$

**1.25 Claim.** Assume  $M = \text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ ,  $I$  is  $\kappa^+$ -wide,  $\kappa < \lambda$  and  $\text{LS}(\mathfrak{K}) + \theta \leq \kappa < \partial$ .

1) Assume that  $\mathcal{L} = \mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]$  and  $\bar{a}_\alpha = \langle \sigma_i(\dots, a_{t(\alpha, i, \ell)}, \dots)_{\ell < n_i} : i < \theta \rangle$  for  $\alpha < \partial$  so  $\sigma_i$  is a  $\tau(\Phi)$ -term, and  $\text{cf}(\partial) > \kappa$ . Assume further that letting  $\bar{t}_\alpha = \langle t(\alpha, i, \ell) : i < \theta, \ell < n_i \rangle$ , the sequence  $\langle \bar{t}_\alpha : \alpha < \partial \rangle$  is indiscernible in  $I$  for quantifier free formulas (i.e. the truth values of  $t(\alpha_1, i_1, \ell_1) < t(\alpha_2, i_2, \ell_2)$  depends only on  $i_1, \ell_1, i_2, \ell_2$  and the truth value of  $\alpha_1 < \alpha_2, \alpha_1 = \alpha_2, \alpha_1 > \alpha_2$ ). Then  $\langle \bar{a}_\alpha : \alpha < \partial \rangle$  is  $(\mathcal{L}, \partial, \kappa)$ -convergent in the model  $M$ .

2) In part (1), even dropping the assumption  $\text{cf}(\partial) > \kappa$ , moreover, the sequence  $\langle \bar{a}_\alpha : \alpha < \partial \rangle$  is  $(\mathcal{L}, \kappa^+, \kappa)$ -convergent and  $(\mathcal{L}, < \omega)$ -indiscernible in  $M$ .

3) In part (1) and in part (2), letting  $J_0 = \{t(0, i, \ell) : t(0, i, \ell) = t(1, i, \ell) \text{ and } i < \theta, \ell < n_i\}$  assume  $J_0 \subseteq J \subseteq I$ ,  $J$  is  $\kappa^+$ -wide (e.g.  $J = \{t(\alpha, i, \ell) : \alpha < \kappa^+, i < \theta, \ell < n_i\}$ ) and  $B$  is the universe of  $\text{EM}_{\tau(\mathfrak{K})}(J, \Phi)$  and  $(i_1, i_2 < \theta, \ell_1 < n_{\ell_1}, \ell_2 < n_{\ell_2}$  and  $[\alpha, \beta < \partial \Rightarrow t(\alpha, i_1, \ell_1) <_I t(\beta, i_2, \ell_2)] \Rightarrow \exists s \in J_0 [\alpha, \beta < \partial \Rightarrow t(\alpha, i_1, \ell_1) <_I t <_I t(\beta, i_2, \ell_2)]$  then  $B$  is a  $(\partial, \kappa)$ -base of  $\{\bar{a}_\alpha : \alpha < \partial\}$ .

4) If  $I$  is well ordered (or just is  $\text{EM}_{\{<\}}(J, \Psi)$ ,  $\Psi \in \Upsilon^{\text{or}}$ ,  $J$  well ordered),  $\text{LS}(\mathfrak{K}) + \theta \leq \kappa, 2^\kappa < \partial, (\forall \alpha < \partial)[|\alpha|^\theta < \partial = \text{cf}(\partial)]$  and  $\bar{b}_\alpha \in {}^\theta M$  for  $\alpha < \partial$ , then for some stationary  $S \subseteq \{\delta < \partial : \text{cf}(\delta) \geq \theta^+\}$ , the sequence  $\langle \bar{b}_\alpha : \alpha \in S \rangle$  is as in part (1) hence is  $(\kappa^+, \kappa)$ -convergent in  $M$ . Moreover, if  $S_0 \subseteq \{\delta < \partial : \text{cf}(\delta) \geq \theta^+\}$  is stationary we can demand  $S \subseteq S_0$ .

5) If in (4) we omit the assumption “ $I$  is well ordered”, and add  $\partial \rightarrow (\partial_1)_{2^\kappa}^2$ , e.g.  $\partial_1 = (2^\kappa)^+$ ,  $\partial = (2^{2^\kappa})^+$  then we can find  $S \subseteq \partial$ ,  $|S| = \partial_1$  such that  $\langle \bar{a}_\alpha : \alpha \in S \rangle$  is as in (1).

*Remark.* In fact the well order case always applies at least if  $\partial < \mu$ .

*Proof.* 1) Let  $\bar{b} \in {}^\kappa M$ , so  $\bar{b} = \langle \sigma_j^*(\dots, a_{s(j,\ell)}, \dots)_{\ell < m_j} : j < \kappa \rangle$  where  $\sigma_i^*$  is a  $\tau(\Phi)$ -term,  $s(j, \ell) \in I$  and let  $\bar{s} = \langle s(j, \ell) : \ell < m_j, j < \kappa \rangle$ .

Now for each  $i_1 < \theta$ ,  $\ell_1 < n_{i_1}$  and  $j_1 < \kappa$ ,  $k_1 < m_{j_1}$  the sequence  $\langle t(\alpha, i_1, \ell_1) : \alpha < \partial \rangle$  is monotonic (in  $I$ ) hence there is  $\alpha(i_1, \ell_1, j_1, k_1) < \partial$  such that

(\*)<sub>1</sub> if  $\beta, \gamma \in \partial \setminus \{\alpha(i_1, \ell_1, j_1, k_1)\}$  and  $\beta < \alpha(i_1, \ell_1, j_1, k_1) \equiv \gamma < \alpha(i_1, \ell_1, j_1, k_1)$  then  $(t(\beta, i_1, \ell_1) <_I s(j_1, k_1)) \equiv (t(\gamma, i_1, \ell_1) <_I s(j_1, k_1))$  and  $(t(\beta, i_1, \ell_1) >_I s(j_1, k_1)) \equiv (t(\gamma, i_1, \ell_1) >_I s(j_1, k_1))$ .

Let  $u := \{\alpha(i_1, \ell_1, j_1, k_1) : i_1 < \theta, \ell_1 < n_{i_1}, j_1 < \kappa, k_1 < m_{j_1}\}$ , it is a subset of  $\partial$  of cardinality  $\leq \theta + \kappa = \kappa$ .

Hence

(\*)<sub>2</sub> if  $\beta, \gamma \in \partial \setminus u$  and  $\beta \mathcal{E}_u \gamma$  which is defined by  $(\forall \alpha \in u)(\alpha < \beta \equiv \alpha < \gamma)$  then  $\bar{t}_\beta \hat{\ } \bar{s}, \bar{t}_\gamma \hat{\ } \bar{s}$  realizes the same quantifier free type in  $I$

Now by clause (c) of 1.14 recalling  $I$  is  $\kappa^+$ -wide we have

(\*)<sub>3</sub> if  $\beta, \gamma \in \partial \setminus u$  and  $\beta \mathcal{E}_u \gamma$  then  $\bar{a}_\beta \hat{\ } \bar{b}, \bar{a}_\gamma \hat{\ } \bar{b}$  realizes the same  $\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]$ -type in  $M$ .

As  $\bar{b}$  was any member of  ${}^\kappa M$  we have gotten

(\*)<sub>4</sub> if  $\bar{b} \in {}^{\kappa \geq} M$ , then for some  $u = u_{\bar{b}} \subseteq \partial$  of cardinality  $\leq \kappa$  we have:  
if  $\beta, \gamma \in \partial \setminus u$  and  $\beta \mathcal{E}_u \gamma$  then  $\bar{a}_\beta \hat{\ } \bar{b}, \bar{a}_\gamma \hat{\ } \bar{b}$  realize the same  $\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]$ -type in  $M$ .

As we are assuming  $\text{cf}(\partial) > \kappa (\geq \theta + \text{LS}(\mathfrak{K}) \geq |\tau_\Phi|)$  we can conclude that

(\*)<sub>5</sub>  $\langle \bar{a}_\alpha : \alpha < \partial \rangle$  is  $(\mathcal{L}, \partial, \kappa)$ -convergent in  $M$ .

So we have proved 1.25(1).

2) We start as in the proof of part (1). However, after (\*)<sub>3</sub> above letting for simplicity  $u^+ = \{\alpha < \partial : \text{for some } \beta \in u \cap \alpha \text{ we have } \alpha + \kappa = \beta + \kappa\}$  we have

(\*)<sub>6</sub> if  $\beta, \gamma \in \partial \setminus u^+$  and  $\beta < \gamma, \neg(\beta E_{u^+} \gamma)$  then we can find  $(\mu^+, I^+, \bar{s}', \bar{b})$  such that

$$(\alpha) \quad I \subseteq I^+ \in K^{\text{lin}}$$

$$(\beta) \quad M^+ = \text{EM}_{\tau(\mathfrak{K})}(I, \Phi) \text{ hence } M \prec_{\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]} N$$

$$(\gamma) \quad \bar{s} = \langle s'(j, k) : k < m, j < \kappa \rangle \text{ a sequence of elements of } I^+$$

$$(\delta) \quad \bar{b}' = \langle \sigma_j^*(\dots, a_{s'(j, \ell)}, \dots)_{\ell < m_j} : j < \kappa \rangle \in {}^\kappa(M^+)$$

$$(\varepsilon) \quad \bar{b} \hat{\ } \bar{a}_\gamma, \bar{b}' \hat{\ } \bar{a}_\gamma \text{ realize the same } \mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]\text{-types in } M^+ \text{ as } \bar{b} \hat{\ } \bar{a}_\gamma, \bar{b} \hat{\ } \bar{a}_\beta \text{ respectively}$$

$$(\zeta) \quad \bar{s} \hat{\ } \bar{t}_\beta, \bar{s}' \hat{\ } \bar{t}_\beta \text{ form a } \Delta\text{-system pair, i.e. are as in } \boxtimes \text{ from 1.5(2).}$$

[Why?

Let  $w^+ = \{(j, k) : k < m_j \text{ and } j < \kappa \text{ and for some } \ell < n_{i_1}, i_1 < \theta \text{ we have } \alpha(i_1, \ell_1, j, k) \in (\beta, \gamma)\}$

$$w^- := \{(j, k) : j < \kappa, k < m_j \text{ and } (j, \kappa) \notin w^+\}.$$

We choose  $I^+$  extending  $I$  and  $\bar{s}_\varepsilon = \langle s_i(j, k) : k < m_j, j < \kappa \rangle$  for  $\varepsilon < \kappa$  such that

- (a) the set of elements of  $I^+$  is the disjoint union of  $I$  and  $\{s_\varepsilon(j, k) : (j, k) \in w \text{ and } \varepsilon \in (0, \kappa)\}$
- (b)  $\bar{s}_\varepsilon, \bar{s}$  realize the same quantifier-free type in  $I^+$
- (c) if  $\varepsilon, \zeta < \kappa$  then  $\bar{t}_{\gamma+\varepsilon} \hat{\ } \bar{s}_\zeta$  realizes in  $I^+$  the quantifier-free type  $\text{tp}_{\text{qf}}(\bar{t}_\beta \hat{\ } \bar{s}, \emptyset, I)$  if  $\varepsilon < \zeta$  and  $\text{tp}_q(\bar{t}_\gamma \hat{\ } \bar{s}, \emptyset, I)$  if  $\varepsilon \geq \zeta$
- (d)  $\langle \bar{t}_{\gamma+\varepsilon} \hat{\ } \bar{s}_\varepsilon : \varepsilon < \kappa \rangle$  is indiscernible for quantifier-free formulas on  $I^+$
- (e)  $\bar{s}_0 = \bar{s}$ .

This is straight. Using  $\bar{s}' = \bar{s}_1$  we are done.]

Now as  $\Phi$  has the  $\kappa$ -non-order property (by Claim 1.5(2) which contains a definition, noting that the assumption of 1.5 holds by 1.18(1) and also 1.18(2)), repeating (\*)<sub>4</sub>, (\*)<sub>5</sub> we get

(\*)<sub>7</sub> for every  $\bar{b} \in {}^\kappa M$ , for some  $u = u_b^+ \in [\partial]^{\leq \kappa}$  if  $\beta, \gamma \in \partial \setminus u^+$  then  $\bar{a}_\beta \hat{\ } \bar{b}, \bar{a}_\gamma \hat{\ } \bar{b}$  realizes the same  $\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]$ -type in  $M$ .

In other words

(\*)<sub>8</sub> the sequence  $\langle \bar{a}_\alpha : \alpha < \partial \rangle$  is  $(\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}], \kappa^+)$ -convergent.

The proof that it is a  $(\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}], < \omega)$ -indiscernible set is similar.

3) Not used; easy by 1.23(2) and convergence. [That is, note that we can find  $I^+$  and  $\bar{a}'_\alpha = \langle \sigma_i(\dots, a_{t'(\alpha, i, \ell)}, \dots)_{\ell_i < n_i} : i < \theta \rangle$  for  $\alpha < \partial + \gamma$  such that:

- (a)  $I^+ \in K^{\text{lin}}$  extend  $I$
- (b)  $t'(\alpha, i, \ell) \in I^+$
- (c)  $\bar{t}'_\alpha = \langle t'(\alpha, i, \ell) : i < \theta, \ell < n_i \rangle$
- (d)  $\langle \bar{t}'_\alpha : \alpha < \partial + \gamma \rangle$  is indiscernible for quantifier-free formulas in  $I^+$
- (e)  $\langle \bar{t}_\alpha : \alpha < \partial \rangle \wedge \langle \bar{t}'_\alpha : \alpha \in [\partial, \partial + \partial] \rangle$  is indiscernible for quantifier-free formulas in  $I'$
- (f) for each  $i < \theta, \ell < n_i$  such that  $t(0, i, \ell) = (j, i, t)$  the convex hull  $I_*$  of  $\{t'(\alpha, i, \ell) : \alpha < \partial\}$  in  $I^+$  is disjoint to  $I$  and if  $s_1 <_I s_2$  and  $(s_1, s_2)_{I^*} \cap I_* = \emptyset$  then  $[s_1, s_2]_{I^*} \cap J_0 \neq \emptyset$ .

So we can average over  $\langle \bar{a}'_\alpha : \alpha < \partial \rangle$  instead averaging over  $\langle \bar{a}_\alpha : \alpha < \partial \rangle$ , and this implies the result. In fact we can weaken the assumption.]

4) Should be clear. [Still let  $\bar{t}_\alpha = \langle t_{\alpha, i} : i < \theta \rangle$  be such that

$\bar{b}_\alpha = \langle \sigma_{\alpha, j}(\dots, a_{t_{\alpha, i(j, \alpha, \ell)}})_{\ell < n(\alpha, j)} : j < \theta \rangle$ . So as  $(\text{LS}(\mathfrak{K}) + |\tau_\Phi|)^\theta < \partial = \text{cf}(\partial)$  for some stationary  $S_1 \subseteq \{\delta < \partial : \text{cf}(\delta) \geq \theta^+\}$  we have  $\alpha \in S_1 \wedge j < \theta \Rightarrow \sigma_{\alpha, j} = \sigma_j$  (hence  $j < \theta \Rightarrow n(\alpha, j) = n(j)$ ) and  $\alpha \in S_1 \wedge j < \theta \wedge \ell < n(j) \Rightarrow i(j, \alpha, \ell) = i(j, \ell)$  and for every  $i_1, i_2 < \theta$  we have  $t_{\alpha, i_1} <_I t_{\alpha, i_2} \equiv (i_1, i_2) \in W$  for some sequence  $\bar{\sigma} = \langle \sigma_j : j < \theta \rangle$  of  $\tau_\Phi$ -terms and  $W \subseteq \kappa \times \kappa$  and sequence  $\langle \langle i(j, \ell) : \ell < n(j) \rangle : j < \theta \rangle$ .

If  $I$  is well ordered, for  $\delta \in S_1$  let  $\gamma_\delta = \text{Min}\{\gamma : \text{if } i < \theta \text{ and there are } \beta < \delta, j < \theta \text{ such that } t_{\delta, i} <_I t_{\beta, j} \text{ and then letting } (\beta_{\delta, i}, j_{\delta, i}) \text{ be such a pair with } t_{\beta_{\delta, i}, j_{\delta, i}} \text{ being } <_I\text{-minimals, we have } \beta_{\delta, i} < \gamma\}$ ; clearly  $\gamma_\delta$  is well defined and  $< \delta$  so by Fodor lemma for some  $\gamma_* < \partial$  the set  $S_1 := \{\delta \in S_2 : \gamma_\delta = \gamma_*\}$  is stationary. As  $|\gamma_*|^\theta < \partial$ , for some  $u \subseteq \theta$  and stationary  $S_3 \subseteq S_2$  we have: if  $\delta \in S_3$  then  $j \in u \Leftrightarrow (\beta_{\delta, i}, j_{\delta, i})$  well defined and  $j \in u \wedge \alpha \in S_3 \Rightarrow (\beta_{\delta, i}, j_{\delta, i}) = (\beta_i, j_i)$  and for each  $i \in u$  the truth value of “ $t_{\delta, i} = t_{\beta_i, j_i}$ ” is the same for all  $\delta \in S_3$ .

Now apply part (1) to  $\langle \bar{b}_\alpha : \alpha \in S_3 \rangle$ .

5) By (1) and the definition of  $\partial \rightarrow (\partial_1)_{2^\kappa}^2$ .

□<sub>1.25</sub>

**1.26 Claim.** 1) If  $M \leq_{\mathfrak{K}} N$  are from  $K_\lambda^*$  and  $\kappa \in [\text{LS}(\mathfrak{K}), \lambda), \kappa^+ < \partial = \text{cf}(\partial) < \lambda$  and moreover  $\theta \leq \kappa$  and  $\bar{a} \in {}^\theta N$  then there is a  $(\kappa^+, \kappa)$ -convergent set  $\mathbf{I} \subseteq {}^\theta M$  of cardinality  $\partial$  such that  $\text{Av}_\kappa(\mathbf{I}, M)$  is realized in  $N$  by  $\bar{a}$ .

2) In fact we can weaken  $M, N \in K_\lambda^*$  to  $M, N \in K_{\geq \beth_{1,1}(\partial')}^*$  where, e.g.  $\partial' = \beth_5(\kappa)^+$ .

3) Assume  $\theta \leq \kappa, \kappa \in [\text{LS}(\mathfrak{K}), \lambda), \partial' = \beth_5(\kappa)^+$  and  $M_1 \in K_{\geq \beth_{1,1}(\partial')}^*$ . Assume further  $M_1 \leq_{\mathfrak{K}} M_2 = \text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$ ,  $|\xi| = \theta$  and  $\mathbf{I} \subseteq {}^\xi(M_1)$  is a  $(\kappa^+, \kappa)$ -convergent set (in  $M_1$ ) of cardinality  $\partial'$ . If  $I_2 <_{K^{\text{fin}}}^* I_3$  (or just  $I_3$  is  $\kappa^+$ -wide over  $I_2$ , which follows as  $|I_2| \geq |\mathbf{I}| = \partial'$ ) and  $M_3 = \text{EM}_{\tau(\mathfrak{K})}(I_3, M_3)$  then



- (a) we can find  $\bar{d} \in {}^\xi(M_3)$  realizing  $\text{Av}_\kappa(M_2, \mathbf{I})$  so well defined
- (b) if  $M_1 \leq_{\mathfrak{K}} N \in K^*$  and  $\bar{d}^* \in {}^\xi N, |\xi| \leq \theta$  then we can find  $\bar{d} \in {}^\xi(M_3)$  realizing  $\text{tp}_{\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]}(\bar{d}^*, M_1, N)$  and  $\text{tp}_{\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]}(\bar{d}, M_2, M_3)$  is the average of some  $(\kappa^+, \kappa)$ -convergent  $\mathbf{I}' \subseteq {}^\alpha(M_1)$  of cardinality  $\partial'$ .

*Remark.* The exact value of  $\partial'$  have no influences for our purpose.

*Proof.* 1) Without loss of generality  $M = \text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ . Let  $\partial_0 = \partial$  and  $\partial_{\ell+1} = \beth_2(\partial_\ell)^+$  for  $\ell = 0, 1$  so  $\partial_\ell < \lambda$  and  $\ell = 1, 2 \Rightarrow (\forall \alpha < \partial_\ell)(|\alpha|^{\kappa+\theta} < \partial_\ell = \text{cf}(\partial_\ell) < \lambda)$  (if  $I$  is well ordered (which is O.K. by 1.19(4)) and  $(\forall \alpha < \partial)(|\alpha|^\kappa < \partial)$  then we can use  $\partial_\ell = \partial$ ).

By 1.24 there is  $B_* \subseteq M$  of cardinality  $< \partial_2$  (or just  $\leq 2^{2^\kappa} < \partial_2$ ) such that  $\text{tp}_{\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]}(\bar{a}, M, N)$  does not  $(\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}], \mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}])$ -split over  $B_*$ .

Now by 1.19(1) for every  $B \subseteq M, |B| < \partial_2$  there is  $\bar{a}' \in {}^\theta M$  realizing in  $M$ , equivalently in  $N$  (with  $\ell g(\bar{x}) = \theta$ , of course), the type  $\text{tp}_{\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]}(\bar{a}, B, N) = \{\varphi(\bar{x}, \bar{b}) : \bar{b} \in {}^\kappa B, \varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}] \text{ and } N \models \varphi[\bar{a}, \bar{b}]\}$ .

We can choose  $J_\alpha, B_\alpha, \bar{a}_\alpha$  by induction on  $\alpha < \partial_2$  such that  $B_\alpha$  includes  $\cup\{\bar{a}_\beta : \beta < \alpha\} \cup B_*$ ,  $B_\alpha$  is the universe of  $\text{EM}(J_\alpha, \Phi)$ ,  $J_\alpha \subseteq I, |J_\alpha| < \partial_2$ ,  $J_\alpha$  increasing with  $\alpha$  and  $J_\alpha$  is quite closed (e.g. is  $\mathfrak{B}_\alpha \cap I$  where  $\mathfrak{B}_\alpha \prec_{\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]} (\mathcal{H}(\chi), \in, <_\chi^*)$  with  $M, N, \text{EM}(I, \Phi), \mathfrak{K}, \langle \bar{a}_\beta : \beta < \alpha \rangle, \mathfrak{K}, \kappa, \theta$  belonging to  $\mathfrak{B}_\alpha$  and  $\mathfrak{B}_\alpha$  has cardinality  $< \partial_2$  and  $\mathfrak{B} \cap \partial_2 \in \partial_2$ ). Then choose  $\bar{a}' = \bar{a}_\alpha$  as above, i.e.  $\bar{a}_\alpha \in {}^\theta M$  realizes the same  $\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]$ -type as  $\bar{a}$  over  $B_\alpha = M \cap \mathfrak{B}_\alpha = \text{EM}_{\tau(\mathfrak{K})}(J_\alpha, \bar{a})$  in  $N$ ; such  $\bar{a}_\alpha$  exists by 1.19(1). So for some set  $S_1 \subseteq \partial_2$  of order type  $\partial_1$  the sequence  $\mathbf{I} = \langle \bar{a}_\beta : \beta \in S_1 \rangle$  is  $(\kappa^+, \kappa)$ -convergent (by 1.25(4),(5)).

It is enough to show that  $\mathbf{I}$  is as required, toward contradiction assume that not. Then there is an appropriate formula  $\varphi(\bar{x}, \bar{y})$  with  $\ell g(\bar{x}) = \theta, \ell g(\bar{y}) = \kappa$  and  $\bar{b} \in {}^\kappa M$  such that  $N \models \varphi[\bar{a}, \bar{b}]$  but  $u := \{\alpha \in S_1 : M \models \varphi[\bar{a}_\alpha, \bar{b}]\}$  has cardinality  $< \kappa^+$ . Now for  $\alpha \in S_1$  as  $J_\alpha$  was chosen “closed enough”, there is  $\bar{b}_\alpha \in {}^\kappa(\text{EM}_{\tau(\mathfrak{K})}(J_\alpha, \Phi)) \subseteq {}^\kappa M$  realizing  $\text{tp}_{\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]}(\bar{b}, B_*, M)$  such that  $\beta \in S_1 \cap \alpha \Rightarrow M \models “\varphi[\bar{a}_\beta, \bar{b}] \equiv \varphi[\bar{a}_\beta, \bar{b}_\alpha]”$  (possible, e.g. as  $|B_\alpha|^{|\mathcal{S} \cap \alpha|} \leq (2^{<\partial_1})^{<\partial_1} < \partial_2$ ).

So, again by 1.25(4),(5), for some  $S_0 \subseteq S_1$  of order type  $\partial = \partial_0$ , the sequence  $\langle \bar{a}_\alpha \hat{\ } \bar{b}_\alpha : \alpha \in S_0 \rangle$  is  $(\mathbb{L}_{\infty, \kappa^+}, \kappa^+, \kappa)$ -convergent in  $M$  and  $(\mathbb{L}_{\infty, \kappa^+}, < \omega)$ -indiscernible. Let  $\alpha \in S_0$  be such that  $|S_0 \cap \alpha| > \kappa$ , possible as  $|S_0| = \partial_0 > \kappa_0$ . So the set  $\{\beta \in S_1 \cap \alpha : M \models \varphi[\bar{a}_\beta, \bar{b}_\alpha]\}$  has cardinality  $\leq \kappa$  (being equal to  $\{\beta \in S_1 \cap \alpha : N \models \varphi[\bar{a}_\beta, \bar{b}]\}$ ) but  $\alpha \in S_0 \subseteq S_1$  and  $|S_0 \cap \alpha| > \kappa$ , so for some  $\beta < \alpha$  from  $S_0, M \models \neg \varphi[\bar{a}_\beta, \bar{b}_\alpha]$  hence by the indiscernibility  $M \models \neg \varphi[\bar{a}_\beta, \bar{b}_\gamma]$  for every  $\beta < \gamma$  from  $S_0$ .

On the other hand if  $\alpha < \beta$  are from  $S_0$  then by the choice of  $\bar{b}_\alpha$  the sequences  $\bar{b}, \bar{b}_\alpha$  realizes the same  $\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]$ -type over  $B_*$ . Now  $\text{tp}_{\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]}(\bar{a}, M, N)$  does not split over  $B_*$  by the choice of  $B_*$  so we have  $N \models \varphi[\bar{a}, \bar{b}] \equiv \varphi[\bar{a}, \bar{b}_\alpha]$  but by the choice of  $\bar{b}$  we have  $N \models \varphi[\bar{a}, \bar{b}]$  hence  $N \models \varphi[\bar{a}, \bar{b}_\alpha]$  hence  $M \models \varphi[\bar{a}_\beta, \bar{b}_\alpha]$  by the choice of  $\bar{a}_\beta$ . Together this contradicts 1.5, i.e., 1.18(1).

2) Similarly (using 1.19(2) instead of 1.19(1)).

3) Clause (a):

By 1.14 and the LS argument (i.e. by 0.17(4)) without loss of generality  $M_1 \in K_{<\lambda}^*$ . Let  $\partial_\ell = \beth_\ell(\kappa)^+$  for  $\ell \leq 5$  so  $\partial' = \partial_5$  and for notational simplicity assume  $\theta \geq \aleph_0$ .

Let  $\{\bar{a}_\alpha : \alpha < \partial'\}$  list the members of  $\mathbf{I}$ , so for each  $\alpha < \partial'$  there is  $I_{2,\alpha} \subseteq I_2$  of cardinality  $\theta$  such that  $\bar{a}_\alpha$  is from  $\text{EM}_{\tau(\mathfrak{K})}(I_{2,\alpha}, \Phi)$ .

For each  $\alpha < \partial'$  let  $\bar{t}^\alpha = \langle t_i^\alpha : i < \theta \rangle$  list  $I_{2,\alpha}$  and so  $\bar{a}_\alpha = \langle \sigma_{\alpha,\zeta}(\bar{t}^\alpha) : \zeta < \xi \rangle$  for some sequence  $\langle \sigma_{\alpha,\zeta}(\bar{x}) : \zeta < \xi \rangle$  of  $\tau_\Phi$ -terms. We can find  $S \subseteq \partial'$  of order type  $\partial_4$  such that  $\zeta < \xi \wedge \alpha \in S \Rightarrow \sigma_{\alpha,\zeta} = \sigma_\zeta$  and  $\langle \bar{t}^\alpha : \alpha \in S \rangle$  is an indiscernible sequence (for quantifier free formulas, in  $I_2$ , of course).

By renaming  $\kappa^+ \subseteq S$ . We define a partition  $\langle u_{-1}, u_0, u_1 \rangle$  of  $\xi$  by

$$u_0 = \{i < \theta : t_i^\alpha = t_i^\beta \text{ for } \alpha, \beta \in S\}$$

$$u_1 = \{i < \theta : t_i^\alpha <_{I_2} t_i^\beta \text{ for } \alpha < \beta \text{ from } S\}$$

$$u_{-1} = \{i < \theta : t_i^\beta <_{I_2} t_i^\alpha \text{ for } \alpha < \beta \text{ from } S\}.$$

We define an equivalence relation  $e$  on  $u_{-1} \cup u_1$

$$\odot \quad i_1 e i_2 \text{ iff for some } \ell \in \{1, -1\}, i_1, i_2 \in u_\ell \text{ and } (t_{i_1}^\alpha <_I t_{i_2}^\beta) \equiv (t_{i_2}^\alpha <_I t_{i_1}^\beta) \text{ for every (equivalently some) } \alpha < \beta \text{ from } S.$$

There is a natural set of representatives:  $W = \{\zeta < \theta : \zeta \in u_{-1} \cup u_1 \text{ and } \zeta = \min(\zeta/e)\}$ .

We now define a linear order  $I_2^+$ ; its set of elements is  $\{t : t \in I_2\} \cup \{t_i^* : i \in u_{-1} \cup u_1\}$  where, of course,  $t_i^* \in I_2^+$  are pairwise distinct and  $\notin I_2$ . The order is defined by (or see  $\otimes_2$  and think)

$$\otimes_1 \quad s_1 <_{I_2^+} s_2 \text{ iff}$$

- (a)  $s_1, s_2 \in I_2$  and  $s_1 <_{I_2} s_2$
- (b)  $s_1 \in I_2, s_2 = t_i^*$  and  $s_1 <_{I_2} t_i^\alpha$  for every  $\alpha < \kappa^+$  large enough
- (c)  $s_1 = t_i^*, s_2 \in I_2$  and  $t_i^\alpha <_{I_2} s_2$  for every  $\alpha < \kappa^+$  large enough
- (d)  $s_1 = t_i^*, s_2 = t_j^*$  and  $t_i^\alpha <_I t_j^\alpha$  for every  $\alpha < \kappa^+$ .

Let  $t_i^* = t_i^\alpha$  for  $i \in u_0$  and any  $\alpha < \kappa^+$ . Let  $M_2^+ = \text{EM}_{\tau(\mathfrak{K})}(I_2^+, \Phi)$ .

It is easy to check (by 1.14(a),(c)) that

- ⊗<sub>2</sub> (a)  $I_2 \subseteq I_2^+$
- (b)  $\bar{t}^* \in \theta(I_2^+)$
- (c) if  $J \subseteq I_2$  has cardinality  $\leq \kappa$  then for every  $\alpha < \kappa^+$  large enough, the sequences  $\bar{t}^*, \bar{t}^\alpha$  realizes the same quantifier free type over  $J$  inside  $I_2^+$ .

Let

$$\otimes_3 \bar{d} := \langle \sigma_\zeta(\bar{t}^*) : \zeta < \xi \rangle \in {}^\xi(M_2^+).$$

Recall that  $\|M_2\| < \lambda$  hence  $|I_2| < \lambda$  and  $I_2$  is  $\kappa^+$ -wide having cardinality  $\geq \partial' > 2^\kappa$ .

Note

$$\otimes_4 \bar{t}^* \text{ realizes } \text{Av}_{\text{qf}}(\{\bar{t}^\alpha : \alpha \in S\}, I_2) \text{ in the linear order } I_2^+.$$

Without loss of generality  $I_2^+ \cap I_3 = I_2$ , so we can find a linear order  $I_4$  of cardinality  $\lambda$  such that  $I_2^+ \subseteq I_4 \wedge I_3 \subseteq I_4$ . As  $I_3$  is  $\kappa^+$ -wide over  $I_2$  (see the assumption and Definition 0.14(6)+(3)), there is a convex subset  $I_3'$  of  $I_3$  disjoint to  $I_2$  which contains a monotonic sequence  $\langle s_\alpha : \alpha < \kappa^+ \rangle$ . Without loss of generality there are elements  $s_\alpha$  ( $\alpha \in [\kappa^+, \lambda \times \kappa^+)$ ) in  $I_4$  such that  $\langle s_\alpha : \alpha < \lambda \times \kappa^+ \rangle$  is monotonic (in  $I_4$ ), and its convex hull is disjoint to  $I_2$ . Let  $I_3^- = I_2 \cup \{s_\alpha : \alpha < \kappa^+\}$  and  $I_3^\pm = I_2 \cup \{s_\alpha : \alpha < \lambda \times \kappa^+\}$ .

Now we use 1.14 several times. First,  $\text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi) \prec_{\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]} \text{EM}_{\tau(\mathfrak{K})}(I_2^+, \Phi) \prec_{\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]} \text{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$  as  $I_2 \subseteq I_2^+ \subseteq I_4$  are  $\kappa^+$ -wide, hence by  $\otimes_4$  the sequence  $\bar{d}$  realizes  $q := \text{Av}_\kappa(\{\langle \sigma_\zeta(\bar{t}^\alpha) : \zeta < \theta \rangle : \alpha < \kappa^+\}, M_2) = \text{Av}(\{\bar{a}_\alpha : \alpha < \kappa^+\}, M_2) = \text{Av}_\kappa(\mathbf{I}, M_2)$  in  $M_2^+$  and also in  $\text{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$ . Second, as  $|I_2| < \lambda$ ,  $I_2 \subseteq I_3^\pm \subseteq I_4$  and  $|I_3^\pm| = |I_4| = \lambda$ , by 1.19(1) we have  $\text{EM}_{\tau(\mathfrak{K})}(I_3^\pm, \Phi) \prec_{\mathbb{L}_{\infty, \lambda}[\mathfrak{K}]} \text{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$  so some  $\bar{d}' \in {}^\xi(\text{EM}_{\tau(\mathfrak{K})}(I_3^\pm, \Phi))$  realizes the type  $q$  in  $\text{EM}_{\tau(\mathfrak{K})}(I_3^\pm, \Phi)$ . Let  $w_1 \subseteq \lambda \times \kappa^+$  be of cardinality  $\leq \theta \leq \kappa$  such that  $\bar{d}'$  belongs to  $\text{EM}_{\tau(\mathfrak{K})}(I_2 \cup \{s_\alpha : \alpha \in w_1\}, \Phi)$ . Choose  $w_2 \subseteq \lambda \times \kappa^+$  of order type  $\kappa^+$  including  $w_1$ , so  $\text{EM}_{\tau(\mathfrak{K})}(I_2 \cup \{s_\alpha : \alpha \in w_2\}, \Phi) \prec_{\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]} \text{EM}_{\tau(\mathfrak{K})}(I_3^\pm, \Phi)$  and  $\bar{d}'$  belongs to the former hence realizes  $q$  in it. But there is an isomorphism  $h$  from  $I_2 \cup \{s_\alpha : \alpha \in w_2\}$  onto  $I_3^-$  over  $I_2$ , hence it induces an isomorphism  $\hat{h}$  from  $\text{EM}_{\tau(\mathfrak{K})}(I_2 \cup \{s_\alpha : \alpha \in w_2\}, \Phi)$  onto  $\text{EM}_{\tau(\mathfrak{K})}(I_3^-, \Phi)$  so  $\hat{h}(\bar{d}')$  realizes  $q$  in the latter. But  $I_3^- \subseteq I_3$  are both  $\kappa^+$ -wide hence by 1.14 the sequence  $\hat{h}(\bar{d}')$  realizes  $q$  in  $M_3 = \text{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$  as required.

Clause (b):

By part (2) we can find appropriate  $\mathbf{I}$  and then apply clause (a).  $\square_{1.26}$

*1.27 Remark.* 1) In fact in 1.24, we can choose  $B$  of cardinality  $\kappa$ , hence similarly in the proof of 1.26(1).

2) Also using solvability to get well ordered  $I$  we can prove : if  $A \subseteq M = \text{EM}_{\tau(\mathfrak{K})}(\lambda, \Phi)$  and  $|A| < \lambda$  then the set of  $\mathbb{L}_{\infty, \kappa^+}[\mathfrak{K}]$ -types realized in  $M$  over  $A$  is  $\leq (|A| + 2)^\kappa$ .

**1.28 Claim.** 1) If  $M \in K_{\geq \kappa}^{**}$  and  $\text{LS}(\mathfrak{K}) \leq \theta$  and  $\partial = \beth_{1,1}(\theta) \leq \kappa \leq \lambda$ , then for  $\bar{a}, \bar{b} \in {}^\theta M$  the following are equivalent: (the difference is using  $\partial$  or  $\kappa$ )

- (a)  $\bar{a}, \bar{b}$  realize the same  $\mathbb{L}_{\infty, \partial}[\mathfrak{K}]$ -type in  $M$
- (b)  $\bar{a}, \bar{b}$  realize the same  $\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]$ -type in  $M$ .

2) For  $M, \theta, \partial, \kappa$  as above, the number of  $\mathbb{L}_{\infty, \partial}[\mathfrak{K}]$ -types of  $\bar{a} \in {}^\theta M$  where  $M = \text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ ,  $|I| \geq \partial$  is  $\leq 2^\theta$ .

*Remark.* Part (1) improves 1.19(3).

*Proof.* 1) Clearly (b)  $\Rightarrow$  (a), so assume clause (a) holds. As  $M \in K_{\geq \kappa}^{**}$  without loss of generality there is a  $\kappa$ -wide linear order  $I$  such that  $M = \text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ ; hence for some  $J \subseteq I$ ,  $|J| = \theta$  we have  $\bar{a}, \bar{b} \in {}^\theta(\text{EM}_{\tau(\mathfrak{K})}(J, \Phi))$ . So for every  $\alpha < (2^\theta)^+$ , by the hence and forth argument for  $\mathbb{L}_{\infty, \beth_\alpha^+}[\mathfrak{K}]$  there are  $J_\alpha, f_\alpha$  such that  $J \subseteq J_\alpha \subseteq I$ ,  $|J_\alpha| = \beth_\alpha$  and  $f_\alpha$  is an automorphism of  $\text{EM}_{\tau(\mathfrak{K})}(J_\alpha, \Phi)$  which maps  $\bar{a}$  to  $\bar{b}$ . Hence as in the proof of 1.19 there is a linear order  $J^+$  of cardinality  $\mu$  extending  $J$  and an automorphism  $f$  of  $M^+ = \text{EM}_{\tau(\mathfrak{K})}(I^+, M)$  mapping  $\bar{a}$  to  $\bar{b}$ . By clause (b) of Claim 1.14 we are done.

2) Easy by clause (c) of 1.14, i.e., by 1.18.  $\square_{1.28}$

**1.29 Claim.** Assume:

- (a)  $I_1 \subseteq I_2, I_1 \neq I_2$ , moreover  $I_1 <_{K^{\text{flin}}} I_2$ , see Definition 0.14(6)
- (b)  $M_\ell = \text{EM}_{\tau(\mathfrak{K})}(I_\ell, \Phi)$  for  $\ell = 1, 2$
- (c)  $\bar{b}, \bar{c} \in {}^\alpha(M_2)$
- (d)  $\theta \geq |\alpha| + \text{LS}(\mathfrak{K})$
- (e)  $\kappa = \beth_{1,1}(\theta_2) \leq \lambda$  where  $\theta_1 = 2^\theta, \theta_2 = (2^{\theta_1})^+$
- (f)  $|I_1| \geq \kappa$
- (g)  $M_1 \leq_{\mathfrak{K}} M_2$ , follows from (a) + (b)

- 1) Assume that for every  $\bar{a} \in {}^{\kappa}>(M_1)$  the sequences  $\bar{a} \hat{\ } \bar{b}, \bar{a} \hat{\ } \bar{c}$  realize the same  $\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]$ -type in  $M_2$ . Then there are  $I_3, M_3$  and  $f$  such that  $I_2 \leq_{K^{\text{flin}}} I_3 \in K_{\lambda}^{\text{flin}}, M_3 = \text{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$  and  $f$  an automorphism of  $M_3$  over  $M_1$  mapping  $\bar{b}$  to  $\bar{c}$ .
- 2) Assume that for every  $\bar{a} \in {}^{\kappa}>(M_1)$  the sequences  $\bar{a} \hat{\ } \bar{b}, \bar{a} \hat{\ } \bar{c}$  realize the same  $\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]$ -type in  $M_2$  (as in part (a)) and  $\beth_{1,1}(\partial) \leq |I_1|$  and  $\partial < \lambda$ . Then for every  $\bar{a} \in {}^{\kappa}>(M_1)$ , the sequences  $\bar{a} \hat{\ } \bar{b}, \bar{a} \hat{\ } \bar{c}$  realize the same  $\mathbb{L}_{\infty, \partial}[\mathfrak{K}]$ -type in  $M_2$ .
- 3) Assume that  $\text{cf}(\lambda) = \aleph_0$  and  $|I_1| = \lambda$  and recall  $\lambda = \beth_{\lambda} > \text{LS}(\mathfrak{K})$ . If  $M_1 \leq_{\mathfrak{K}} M_2^* \in K_{\lambda}^*$  then for some  $I_3$ , a linear order  $\leq_{K_{\lambda}^{\text{flin}}}$ -extending  $I_2$  the model  $M_2^*$  can be  $\leq_{\mathfrak{K}}$ -embedded into  $M_3 := \text{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$  over  $M_1$ .

*Remark.* 1) Under mild assumptions with somewhat more work in 1.29(1),(3) we can choose  $I_3 = I_2$  (but for this has to be more careful with the linear orders). Recall that for  $I \in K_{\lambda}^{\text{lin}}$  like  $I_2$  in 1.8(c) we have  $\alpha < \lambda^+ \Rightarrow I \times \alpha$  can be embedded into  $I$  and 1.4(1)(d).

*Proof.* 1) There is  $J_2 \subseteq I_2$  of cardinality  $\leq \theta$  such that  $\bar{b}, \bar{c} \in {}^{\alpha}(\text{EM}_{\tau(\mathfrak{K})}(J_2, \Phi))$ ; let  $J_1 = I_1 \cap J_2$ .

We define a two-place relation  $\mathcal{E}$  on  $I_2 \setminus J_2$  :  $s \mathcal{E} t$  iff  $(\forall x \in J_2)(x <_{I_2} s \equiv x <_{I_2} t)$ . Clearly  $\mathcal{E}$  is an equivalence relation. As  $I_1 <_{K^{\text{flin}}} I_2$  clearly

- $\odot_1$  ( $\alpha$ ) any interval of  $I_1$  has cardinality  $|I_1| \geq \kappa$
- ( $\beta$ ) for every  $t \in I_2 \setminus J_2$  the equivalence class  $t/\mathcal{E}$  is a singleton or has  $|I_2| \geq \kappa$  members,
- ( $\gamma$ ) for every  $t \in I_1 \setminus J_1, (t/\mathcal{E}) \cap I_1$  is a singleton or has  $|I_1| \geq \kappa$  members
- ( $\delta$ )  $I_1 \setminus J_2$  has at least  $\kappa$  elements
- ( $\varepsilon$ )  $\mathcal{E}$  has  $\leq 2^{|J_2|} \leq 2^{\theta}$  equivalence classes
- ( $\zeta$ ) we may  $\leq_{K^{\text{flin}}}$ -increase  $I_2$  so without loss of generality
- (\*)<sub>1</sub>  $t \in I_2 \setminus J_2 \Rightarrow |t/\mathcal{E}| = |I_2|$
- (\*)<sub>2</sub> for every  $t \in I_1$  for some  $s_1, s_2 \in I_2$  we have  $s_1 <_{I_2} t <_{I_2} s_2$  and  $(s_1, t_{I_2}), (t, s_2)_{I_2}$  are disjoint to  $I_1$ .

Let  $\langle \mathcal{U}_i : i < i(*) \rangle$  list the equivalence classes of  $\mathcal{E}$ , so without loss of generality  $i(*) \leq 2^{\theta}$ . For  $\ell = 0, 1$  let  $u_{\ell} = \{i < i(*) : \mathcal{U}_i \cap I_1 \text{ has exactly } \ell \text{ members}\}$  and let  $u_2 = i(*) \setminus u_0 \setminus u_1$ , so by clause  $\odot_1(\gamma)$ , i.e. the definition of  $I_1 \in K^{\text{flin}}$  we have  $i \in u_2 \Rightarrow |\mathcal{U}_i \cap I_1| = |I_1| \geq \kappa$ . For  $i \in u_1$  let  $t_i^*$  be the unique member of  $\mathcal{U}_i \cap I_1$ .

Without loss of generality  $u_1 = \{i : i \in [j_0^*, j_1^*]\}$  for some  $j_0^* \leq j_1^* \leq i(*)$  and let  $i'(*) = i(*) + (j_1^* - j_0^*)$  and  $u'_1 = [i(*), i'(*)]$  and define  $\mathcal{U}'_i$  for  $i < i'(*)$  by

- $\odot_2$  (a)  $\mathcal{U}'_i = \mathcal{U}_i$  if  $i \in u_0 \cup u_2$
- (b)  $\mathcal{U}'_i = \{t \in \mathcal{U}_i : t < t_i^*\}$  if  $i \in u_1$  and
- (c)  $\mathcal{U}'_i = \{t \in \mathcal{U}_i : t_i^* <_{I_2} t\}$  if  $i \in [i(*), i'(*)]$ ,  $\iota \in (j_0^*, j_1^*)$  and  $i - i(*) = \iota - j_0^*$ .

For  $i < i'(*)$  let  $\langle t_{i,\alpha} : \alpha < \kappa \rangle$  be a sequence of pairwise distinct members of  $\mathcal{U}'_i$  such that  $i \in u_2 \Rightarrow t_{i,\alpha} \in I_1$  and  $i \in u_0 \Rightarrow t_{i,\alpha} \notin I_1$ , this actually follows. By  $\odot_1(\zeta)$  and  $\odot_1(\beta), (\gamma)$  we can find such  $t_{i,\alpha}$ 's.

For  $\zeta < \theta_2$  (see clause (e) of the assumption so  $\beth_\zeta < \kappa$ ) let  $J_{1,\zeta} = \{t_{i,\alpha} : i \in u_2, \alpha < \beth_\zeta\} \cup J_1 \cup \{t_i^* : i \in u_1\}$ . Now by the hence and forth argument (or see 0.17(2)) for each  $\zeta < \theta_2$ , there are  $J_{2,\zeta}$  and  $f_\zeta$  such that  $J_{2,\zeta} \subseteq I_2$  is of cardinality  $\beth_\zeta$ , it includes  $J_{1,\zeta} \cup J_2$  and also  $\{t_{i,\alpha} : i < i'(*) \text{ and } \alpha < \beth_\zeta\}$  and  $f_\zeta$  is an automorphism of  $\text{EM}_{\tau(\mathfrak{K})}(J_{2,\zeta}, \Phi)$  over  $\text{EM}_{\tau(\mathfrak{K})}(J_{1,\zeta}, \Phi)$  mapping  $\bar{b}$  to  $\bar{c}$ .

(Why? Let  $\bar{a}_0$  list  $\text{EM}(J_{1,\zeta}, \Phi)$  so  $\bar{a}_0 \hat{\ } \bar{b}, \bar{a}_0 \hat{\ } \bar{c}$  realize the same  $\mathbb{L}_{\infty, \beth_\zeta^+}[\mathfrak{K}]$ -type in  $M_2$ , and  $f$  be the mapping taking  $\bar{a}_0 \hat{\ } \bar{b}$  to  $\bar{a}_0 \hat{\ } \bar{c}$ , etc.)

Now we shall immitate the proof of 1.19. By renaming without loss of generality there is a transitive set  $B$  (in the set theoretic sense) of cardinality  $\leq \theta_1 = 2^\theta$  which includes

- $\oplus(a)$   $J_1, J_2$
- (b)  $\Phi$  (i.e.  $\tau_\Phi$  and  $\langle (\text{EM}(n, \Phi), a_\ell)_{\ell < n} : n < \omega \rangle$ )
- (c)  $\mathfrak{K}$ , i.e.,  $\tau_{\mathfrak{K}}$  and  $\{(M, N) : M \leq_{\mathfrak{K}} N \text{ have universe included in } \text{LS}(\mathfrak{K})\}$
- (d)  $\langle t_i^* : i \in u_1 \rangle$  so each  $t_i^*$  for  $i \in u_1$
- (e) the ordinal  $i(*)$ .

Let  $\chi$  be large enough, let  $\mathfrak{B} = (\mathcal{H}(\chi), \in, <_\chi^*)$  and let  $\mathfrak{B}_\zeta^+$  be  $\mathfrak{B}$  expanded by

- $\otimes_1$  (a)  $Q^{\mathfrak{B}_\zeta} = \{\alpha : \alpha < \beth_\zeta\}$
- (b)  $P_i^{\mathfrak{B}_\zeta} = J_{2,\zeta} \cap \mathcal{U}'_i$  for  $i < i'(*)$
- (c)  $F_2^{\mathfrak{B}_\zeta}(t) = a_t$  for  $t \in I_2$
- (d)  $H^{\mathfrak{B}_\zeta} = f_\zeta$  and  $Q_1^{\mathfrak{B}_\zeta} = J_{1,\zeta}, Q_2^{\mathfrak{B}_\zeta} = J_{2,\zeta}$
- (e) for  $i < i'(*)$ ,  $H_i^{\mathfrak{B}_\zeta}$  is the function mapping  $\alpha < \beth_\zeta$  to  $t_{i,\alpha}$
- (f) individual constants for  $B$  and for each  $x \in B$ ,  
hence, e.g. for  $t_i^* (i \in u_1), J_1, J_2, t$  for  $t \in J_2$
- (g) individual constants  $J_{1,*}, J_{2,*}$  interpreted as the linear orders  $J_{1,\zeta}, J_{2,\zeta}$  respectively and individual constants for  $M_\ell^+ = \text{EM}(J_{0,\zeta}, \Phi)$ , and  $\langle a_t : t \in I_\ell \rangle$  for  $\ell = 1, 2$ .

As in the proof of 1.19 there is a  $\tau(\mathfrak{B}^+)$ -model  $\mathfrak{C}$ , such that

- ⊠ (a) for some unbounded  $S \subseteq \theta_2$ 
  - ( $\alpha$ )  $\mathfrak{C}$  is a first order elementarily equivalent to  $\mathfrak{B}_\zeta^+$  for every  $\zeta \in S$
  - ( $\beta$ )  $\mathfrak{C}$  omits every type omitted by  $\mathfrak{B}_\zeta$  for every  $\zeta \in S$ .
- In particular this gives
- ( $\gamma$ )  $\mathfrak{C}$  omits the type  $\{x \neq b \wedge x \in B : b \in B\}$  so
- ( $\delta$ ) without loss of generality  $b \in B \Rightarrow b^\mathfrak{C} = b$

- (b)  $\mathfrak{C}$  is the Skolem hull of some infinite indiscernible sequence  $\langle y_r : r \in I \rangle$ , where  $I$  an infinite linear order and  $y_r \in Q^\mathfrak{C}$  for  $r \in I$ .

Without loss of generality  $I \in K^{\text{flin}}$  and  $I_2$  can be  $\leq_{K^{\text{flin}}}$ -embedded into  $I$  say by the function  $g$  such that  $(\forall t \in I_2)(\exists s_1, s_2 \in I)[s_1 <_I g(t) <_I s_2 \wedge (\forall t' \in I_2)(t' <_{I_2} t \rightarrow g(t') <_I s_1) \wedge (\forall t' \in I_2)(t <_{I_2} t' \rightarrow s_2 <_I g(t'))]$ ; and also  $\|\mathfrak{C}\| = |I|$ . Hence for each  $i < i'(*)$  there is an embedding  $h_i$  of the linear order  $\mathcal{U}_i'$ , i.e.,  $I_2 \upharpoonright \mathcal{U}_i'$  into  $(P_i^\mathfrak{C}, (<_{I_2})^\mathfrak{C})$  such that  $t \in \mathcal{U}_i' \Rightarrow (t \in I_1 \leftrightarrow h_i(t) \in Q_1^\mathfrak{C})$ .

[Why? Case 0:  $i \in u_0$ .

Trivial.

Case 1:  $i \in u_1 \cup u_1'$ .

Similar to Case 0 as  $\mathcal{U}_i' \cap I_1 = \emptyset$ , of course, we take care that  $a = h_i(t) \wedge t \in \mathcal{U}_i' \wedge i \in u_1 \Rightarrow \mathfrak{C} \models "a <_{I_2} t_i^*" and similarly for  $u_{-1}$ .$

Case 2:  $i \in u_2$ .

First approximation is  $h_i' = (H_i^\mathfrak{C} \circ (g \upharpoonright \mathcal{U}_i))$ , so  $t \in \mathcal{U}_i \Rightarrow h_i'(t) \in Q_1^\mathfrak{C}$ . However by the choice of  $g$  we can find  $\langle (s_t^-, s_t^+) : t \in \mathcal{U}_i \rangle$  such that:

- ( $\alpha$ )  $s_t^-, s_t^+ \in Q_2^\mathfrak{C}$
- ( $\beta$ )  $(s_t^-, s_t^+)_{I_2^\mathfrak{C}} \cap Q_2^\mathfrak{C} = \{h_i'(t)\}$ .

As  $I_2$  is dense with no extremal members (being from  $K^{\text{flin}}$ ) clearly  $t_1 <_{I_2 \upharpoonright \mathcal{U}_i'} t_2 \Rightarrow s_{t_1}^+ <_{(I_2)^\mathfrak{C}} s_{t_2}^-$ . Now choose  $h_i$  by:  $h_i(t)$  is  $h_i'(t)$  if  $t \in I_1$  and is  $s_{t_1}^+$  if  $t \in I_1 \setminus I_2$ .

Hence there is an embedding  $h$  of the linear order  $I_2$  into  $J_{1,*}^\mathfrak{C}$  such that:

⊗<sub>2</sub>  $h(t)$  is:

- (a)  $t$  if  $t \in J_2 \cup \{t_i^* : i \in u_1\}$
- (b)  $h_i(t)$  if  $t \in \mathcal{U}_i'$  and  $i < i'(*)$ .

Note

- ⊗<sub>3</sub> for every  $t \in I_2 \setminus J_2$  for some  $i < i(*) \leq \theta_1$  we have  $(\forall s \in J_2)[s <_{I_2} t \equiv s <_{I_2} h_i(t_{i,0})]$

hence by the omitting type demand in  $\boxtimes(a)(\beta)$ :

$$\otimes'_3 \text{ for } t \in I_2^c \setminus J_2 \text{ for some } i < i(*) \text{ we have } (\forall s \in J_2)[s <_{I_2^c} t \equiv s <_{I_2^c} (h_i(t_{i,0}))].$$

We can find a linear order  $I_3, I_2 \subseteq I_3$  and an isomorphism  $h_*$  from  $I_3$  onto  $Q_2^c$  extending  $h$ , so clearly  $I_3 \in K^{\text{flin}}$  and without loss of generality  $h(I_2) <_{K^{\text{flin}}} I_3$ . Now let  $\hat{h}_*$  be the isomorphism which  $h_*$  induces from  $\text{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$  onto  $(\text{EM}_{\tau(\mathfrak{K})}(J_{2,*}^c, \Phi))^c$ , so e.g., it maps for each  $t \in I_2$ , the member  $a_t$  of the skeleton to  $F_2^c(h_*(t))$ .

Note that  $h_*$  maps  $\mathcal{U}_i \cap I_1$  into  $Q_1^c \subseteq I_1^c$  when  $\mathcal{U}_i \subseteq I_1$  and is the identity on  $J_1 \cup \{t_i^* : i \in u_1\}$  so recalling  $Q^{\mathfrak{B}_\zeta} = J_{1,\zeta} = \{t_{i,\alpha} : i \in u_2 \text{ and } \alpha < \beth_\zeta\} \cup J_1 \cup \{t_i^* : i \in u_1\}$  hence it map  $I_1$  into  $Q_1^c$  but  $\mathfrak{B}_\zeta \models "H \text{ is a unary function, an automorphism of } \text{EM}_{\tau(\mathfrak{K})}(J_{2,*}^c, \Phi) \text{ mapping } \bar{b} \text{ to } \bar{c} \text{ and is the identity on } \text{EM}_{\tau(\mathfrak{K})}(J_{1,*}^c, \Phi)"$ . Now  $(\hat{h}_*)^{-1}H^c(\hat{h}_*)$  is an automorphism of  $\text{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$  as required.

2) By part (1), i.e. choose  $I_3, M_3, f_3$  as there; so as  $f$  is an automorphism of  $M_3$  over  $M_1$  mapping  $\bar{b}$  to  $\bar{c}$ , clearly  $\bar{b}, \bar{c}$  realize the same  $\mathbb{L}_{\infty, \partial}[\mathfrak{K}]$ -type over  $M_1$  inside  $M_3$ . The desired result (the type inside  $M_2$  rather than inside  $M_3$ ) follows because  $M_1 \prec_{\mathbb{L}_{\infty, \partial}[\mathfrak{K}]} M_2 \prec_{\mathbb{L}_{\infty, \partial}[\mathfrak{K}]} M_3$  by 1.14(a).

3) Let  $M_2^* = \bigcup_{n < \omega} M_{2,n}^*$  be such that  $n < \omega \Rightarrow M_{2,n}^* \leq_{\mathfrak{K}} M_{2,n+1}^*$  and  $\|M_{2,n}^*\| < \lambda$ .

Let  $\bar{c}_n$  list  $M_{2,n}^*$  for  $n < \omega$  (with no repetitions) and be such that  $\bar{c}_n \triangleleft \bar{c}_{n+1}$ . Let  $\theta_n = \|M_{2,n}^*\| + \text{LS}(\mathfrak{K})$  so without loss of generality  $\theta_n = \ell g(\bar{c}_n)$  and let  $\theta'_n = \beth_3(\theta_n), \kappa_n = \beth_{1,1}(\theta'_n)$ , without loss of generality  $\kappa_n < \theta_{n+1}$  and we choose for each  $n < \omega$ , a sequence  $\bar{b}_n \in \ell g(\bar{c}_n)(M_2)$  realizing  $\text{tp}_{\mathbb{L}_{\infty, \kappa_n^+}[\mathfrak{K}]}(\bar{c}_n, M_1, M_2^*)$  in  $M_2$ . This is possible by 1.26(3) after possibly  $<_{K^{\text{flin}}}$ -increasing  $I_2$ .

Now we choose  $(I_{3,n}, f_n, M_{3,n}, \bar{b}'_n)$  by induction on  $n$  such that

- (\*) (a)  $I_{3,0} = I_2$  and  $I_{3,n} \in K_\lambda^{\text{lin}}$
- (b)  $n = m + 1 \Rightarrow I_{3,m} <_{K^{\text{flin}}} I_{3,n}$
- (c)  $M_{3,n} = \text{EM}_{\tau(\mathfrak{K})}(I_{3,n}, \Phi)$  (hence  $n = m + 1 \Rightarrow M_{3,m} \leq_{\mathfrak{K}_\lambda} M_{3,n}$ )
- (d)  $f_n$  is an automorphism of  $M_{3,n}$  over  $M_1$
- (e)  $\bar{b}'_n \in \ell g(\bar{b}_n)(M_{3,n})$  realizes  $\text{tp}_{\mathbb{L}_{\infty, \kappa_n^+}[\mathfrak{K}]}(\bar{c}_n, M_1, M_2^*)$
- (f) if  $n = m + 1$  then  $\bar{b}'_m \leq \bar{b}'_n$
- (g) if  $n = m + 1$  then  $f_n$  maps  $\bar{b}_{n+1} \upharpoonright \ell g(\bar{b}_n)$  to  $\bar{b}'_n$  and  $f_0$  maps  $\bar{b}_0$  to  $\bar{b}'_0$ .

For  $n = 0, I_{3,0}, M_{3,0}$  are defined in clauses (a),(c) of (\*) and we let  $f_0 = \text{id}_{M_2} = \text{id}_{M_{3,n}}, \bar{b}'_0 = \bar{b}_0$  this is trivially as required. For  $n = m + 1$  we apply part (1) with

$$\boxtimes \quad I_1, I_{3,m}, M_1, M_{3,m}, \bar{b}_{n+1} \upharpoonright \ell g(\bar{c}_m), \bar{b}'_m, \theta_m, \kappa_m \text{ here} \\ \text{standing for } I_1, I_2, M_1, M_2, \bar{b}, \bar{c}, \theta, \kappa \text{ there.}$$



Why its assumptions holds? The main point is to check that for every  $\bar{a} \in {}^{\kappa_m} > (M_1)$  the sequences  $\bar{a}^\wedge(\bar{b}_{n+1} \upharpoonright \theta_m), \bar{a}^\wedge \bar{b}'_m$  realize the same  $\mathbb{L}_{\infty, \kappa_m}[\mathfrak{K}]$ -type in  $M_{3,m}$ . Now  $\bar{a}^\wedge(\bar{b}_{m+1} \upharpoonright \theta_m), \bar{a}^\wedge \bar{b}'_m$  realize the same  $\mathbb{L}_{\infty, \kappa_m}[\mathfrak{K}]$ -type in  $M_{3,m}$  by the induction hypothesis. Also the sequences  $\bar{b}_{n+1} \upharpoonright \theta_m, \bar{b}_{m+1} \upharpoonright \theta_m$  satisfy for any  $\bar{a} \in {}^{\kappa_m} (M_1)$  the sequences  $\bar{a}^\wedge(\bar{b}_{n+1} \upharpoonright \theta_m), \bar{a}^\wedge(\bar{b}_{m+1} \upharpoonright \theta_m)$  realize the same  $\mathbb{L}_{\infty, \kappa_m}[\mathfrak{K}]$ -type in  $M_{3,m}$  because the  $\mathbb{L}_{\infty, \kappa_m}[\mathfrak{K}]$ -type which  $\bar{a}^\wedge(\bar{b}_{n+1} \upharpoonright \theta_m)$  realizes in  $M_{3,m}$  is the same as the  $\mathbb{L}_{\infty, \kappa_m}[\mathfrak{K}]$ -type it realizes in  $M_2 = M_{3,0}$  which (by the choice of  $\bar{b}_{n+1}$ ) is equal to the  $\mathbb{L}_{\infty, \kappa_m}[\mathfrak{K}]$ -type which  $\bar{a}^\wedge(\bar{c}_{n+1} \upharpoonright \theta_m)$  realizes in  $M_2^*$  which is the same as the  $\mathbb{L}_{\infty, \kappa_m}[\mathfrak{K}]$ -type which  $\bar{a}^\wedge(\bar{c}_{m+1} \upharpoonright \theta_m)$  realizes in  $M_2^*$  which is equal to the  $\mathbb{L}_{\infty, \kappa_m}[\mathfrak{K}]$ -type which  $\bar{a}^\wedge(\bar{b}_{m+1} \upharpoonright \theta_m)$  realizes in  $M_{3,m}$ .

By the last two sentences for every  $\bar{a} \in {}^{\kappa_m} > (M_1)$  the sequences  $\bar{a}^\wedge(\bar{b}_{n+1} \upharpoonright \theta_m), \bar{a}^\wedge \bar{b}'_m$  realizes the same  $\mathbb{L}_{\infty, \kappa_m}[\mathfrak{K}]$ -type in  $M_{3,m}$ , so indeed the assumptions of part (1) holds for the case we are trying to apply it, see  $\square$  above.

So we get the conclusion of part (1), i.e. we get  $I_{3,n}, f_n$  here standing for  $I_3, f$  there so  $I_{3,m} <_{K_\lambda^{\text{flin}}} I_{3,n}$  and  $f_n$  is an automorphism of  $M_{3,n} = \text{EM}_{\tau(\mathfrak{K})}(I_{3,n}, \Phi)$  over  $M_1$  mapping  $\bar{b}_{n+1} \upharpoonright \theta_m$  to  $\bar{b}'_m$ . Now we let  $\bar{b}'_n = f_n(\bar{b}_{n+1} \upharpoonright \theta_n)$  and can check all the clauses in (\*). Hence we have carried the induction. So we can satisfy (\*).

So  $\bar{b}'_n$  satisfies the requirements on  $\bar{b}_n$  and  $\bar{b}'_n \triangleleft \bar{b}'_{n+1}$ . Let  $I_3 = \cup \{I_{3,n} : n < \omega\}$  and let  $M_3 = \text{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$  and let  $g : M_2^* \rightarrow M_3$  map  $c_{n,i}$  to  $b'_{n,i}$  for  $i < \ell g(\bar{c}_n), n < \omega$ , easily it is as required. That is,  $g(c_{n,i})$  is well defined as  $c_{n,i} \mapsto b'_{n,i}, (i < \ell g(\bar{c}_n))$  is a well defined mapping for each  $n$  and  $i < \ell g(\bar{c}_n) \Rightarrow c_{n,i} = c_{n+1,i} \wedge b'_{n,i} = b'_{n+1,i}$ . Also  $g \upharpoonright \{c_{n,i} : i < \ell g(\bar{c}_n)\}$  is a  $\leq_{\mathfrak{K}}$ -embedding of  $M_{2,n}^*$  into  $M_3$  and is the identity on  $M_{2,n}^* \cap M_1$  as  $\bar{c}_n$  list the elements of  $M_{2,i}$  and  $\text{tp}_{\mathbb{L}_{\infty, \kappa_n^+}[\mathfrak{K}]}(\bar{c}_n, M_1, M_2^*) = \text{tp}_{\mathbb{L}_{\infty, \kappa_n^+}[\mathfrak{K}]}(\bar{b}'_n, M_1, M_3)$  by clause (e) of (\*). But  $\langle g \upharpoonright M_{2,n}^* : n < \omega \rangle$  is  $\subseteq$ -increasing with union  $g$  so by Ax(V) of a.e.c.  $g$  is a  $\leq_{\mathfrak{K}}$ -embedding of  $M_2^*$  into  $M_3$ . Lastly, obviously  $g \supseteq \cup \{\text{id}_{M_{2,n}^* \cap M_1} : n < \omega\} = \text{id}_{M_1}$ , so we are done.  $\square_{1.29}$

We arrive to the crucial advance:

**1.30 The Amalgamation Theorem.** *If  $\text{cf}(\lambda) = \aleph_0$ , then  $\mathfrak{K}_\lambda^*$ , i.e.,  $(K_\lambda^*, \leq_{\mathfrak{K}} \upharpoonright K_\lambda^*)$  has amalgamation, even disjoint one.*

*Proof.* So assume  $M_0 \leq_{\mathfrak{K}_\lambda^*} M_\ell$  for  $\ell = 1, 2$ . Choose  $I_0 \in K_\lambda^{\text{flin}}$  so  $M'_0 := \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi) \in K_\lambda^*$  but  $K_\lambda^*$  is categorical (see 1.16 or 1.19(4)) hence  $M'_0 \cong M_0$ , so without loss of generality  $M'_0 = M_0$ . Choose  $I_1 \in K_\lambda^{\text{flin}}$  such that  $I_0 <_{K^{\text{flin}}} I_1$  and let  $M'_1 = \text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi)$  so  $M_0 \leq_{\mathfrak{K}} M'_1$ . By applying 1.29(3) with  $I_0, I_1, M_0, M'_1, M_1$  here standing for  $I_1, I_2, M_1, M_2, M_2^*$  there, we can find a pair  $(I_2, f_1)$  such that  $I_1 <_{K_\lambda^{\text{flin}}} I_2$  and  $f_1$  is a  $\leq_{\mathfrak{K}}$ -embedding of  $M_1$  into  $M'_2 := \text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$  over  $M_0$ . Apply 1.29(3) again with  $I_0, I_2, M_0, \text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi), M_2$  here standing for  $I_1, I_2, M_1, M_2, M_2^*$

there. So there is a pair  $(I_3, f_2)$  such that  $I_2 <_{K_\lambda^{\text{flin}}} I_3$  and  $f_2$  is  $\leq_{\mathfrak{K}}$ -embedding  $M_2$  into  $M_3 := \text{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$  over  $M_0 = \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$ . Of course,  $M_3 \in K_\lambda^*$  and we are done proving the “has amalgamation”.

Why disjoint? Let  $(I_4, h)$  be such that  $I_3 <_{K_\lambda^{\text{flin}}} I_4$  and  $h$  is a  $\leq_{K^{\text{flin}}}$ -embedding of  $I_3$  into  $I_4$  over  $I_0$  such that  $h(I_3) \cap I_3 = I_0$ . Now  $h$  induces an isomorphism  $\hat{h}$  from  $\text{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$  onto  $\text{EM}_{\tau(\mathfrak{K})}(h(I_3), \Phi) \leq_{\mathfrak{K}} M_3$ .

Lastly, by our assumptions on  $\Phi$  if  $J_1, J_2 \subseteq J$  and  $I_1 \cap I_2$  is a dense linear order (in particular with neither first nor last member, e.g. are from  $K_\lambda^{\text{flin}}$  as in our case) then  $\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \cap \text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi) = \text{EM}_{\tau(\mathfrak{K})}(I_1 \cap I_2, \Phi)$ . So in particular, above  $\text{EM}_{\tau(\mathfrak{K})}(I_3, \Phi) \cap \text{EM}_{\tau(\mathfrak{K})}(\hat{h}(I_3, \Phi) = \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$  and  $f_1, \hat{h} \circ f_2$  are  $\leq_{\mathfrak{K}}$ -embeddings of  $M_1, M_2$  respectively over  $M_0 = \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$  into  $\text{EM}_{\tau(\mathfrak{K})}(I_3, \Phi) \leq_{\mathfrak{K}} \text{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$  and  $\text{EM}_{\tau(\mathfrak{K})}(h(I_3), \Phi) \leq_{\mathfrak{K}} \text{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$ , respectively, so we are done.  $\square_{1.30}$

**1.31 Claim.** Assume  $\text{cf}(\lambda) = \aleph_0$ . If  $\delta < \lambda^+$ , the sequence  $\langle M_i : i < \delta \rangle$  is  $\leq_{\mathfrak{K}}$ -increasing continuous and  $M_i \in K_\lambda^*$  for  $i < \delta$ , then  $M_\delta := \cup \{M_i : i < \delta\}$  can be  $\leq_{\mathfrak{K}}$ -embedded into some member of  $K_\lambda^*$ .

*Proof.* We choose  $I_i \in K_\lambda^{\text{flin}}$  by induction on  $i \leq \delta$ , which is  $<_{K_\lambda^{\text{flin}}}$ -increasing continuous with  $i$  and a  $\leq_{\mathfrak{K}}$ -embedding  $f_i$  of  $M_i$  into  $N_i := \text{EM}_{\tau(\mathfrak{K})}(I_i, \Phi)$ , increasing continuous with  $i$ . For  $i = 0$  choose  $I_0 \in K_\lambda^{\text{flin}}$ , so  $N_0 := \text{EM}_{\tau(\mathfrak{K})}(I_0, M)$  is isomorphic to  $M_0$  hence  $f_0$  exists; for  $i$  limit use  $I_i := \cup \{I_j : j < i\}$  and  $f_i := \cup \{f_j : j < i\}$ . So assume  $i = j + 1$ . Now we can find  $M'_i, f'_i$  satisfying:  $f'_i$  is an isomorphism from  $M_i$  onto  $M'_i$  extending  $f_j$  such that  $f_j(M_j) \leq_{\mathfrak{K}} M'_i$  (actually this trivially follows) and  $M'_i \cap N_j = f_j(M_j)$ ; so also  $M'_i$  belongs to  $K_\lambda^*$ . Now  $f_j(M_j), \text{EM}_{\tau(\mathfrak{K})}(I_j, \Phi), M'_i$  can be disjointly amalgamated (by 1.30) in  $(K_\lambda^*, \leq_{\mathfrak{K}})$ , so there is  $M_i^* \in K_\lambda^*$  such that  $N_j = \text{EM}_{\tau(\mathfrak{K})}(I_j, \Phi) \leq_{\mathfrak{K}} M_i^*$  and  $M'_i \leq_{\mathfrak{K}} M_i^*$ . Now by 1.29(3) there are  $I_i, g_i$  such that  $I_j <_{K_\lambda^{\text{flin}}} I_i$  and  $g_i$  is a  $\leq_{\mathfrak{K}}$ -embedding of  $M_i^*$  into  $N_i := \text{EM}_{\tau(\mathfrak{K})}(I_i, \Phi)$  over  $\text{EM}_{\tau(\mathfrak{K})}(I_j, \Phi)$ . Let  $f_i = g_i \circ f'_i$ , clearly it is as required. Having carried the induction,  $f_\delta$  is a  $\leq_{\mathfrak{K}}$ -embedding of  $M_\delta$  into  $\text{EM}_{\tau(\mathfrak{K})}(\bigcup_{j < \delta} I_j, \Phi)$ , as promised.  $\square_{1.31}$

**1.32 Claim.** 1) Assume  $\text{cf}(\lambda) = \aleph_0$ . For every  $M_0 \in K_\lambda^*$  there is a  $\leq_{\mathfrak{K}}$ -extension  $M_1 \in K_\lambda^*$  of  $M_0$  such that: if  $M_0 \leq_{\mathfrak{K}_\lambda} M_2 \in K_\lambda^*$  and  $\bar{a} \in {}^{\lambda >} (M_2)$  then for some  $(M_3, f)$  we have:

$M_1 \leq_{\mathfrak{K}} M_3 \in K_\lambda^*, f$  is a  $\leq_{\mathfrak{K}}$ -embedding of  $M_2$  into  $M_3$  over  $M_0$  and  $f(\bar{a}) \in {}^{\lambda >} (M_2)$ .

2) Assume  $\text{cf}(\lambda) = \aleph_0$ . For every  $M_0 \in K_\lambda^*$  there is a  $\leq_{\mathfrak{K}}$ -extension  $M_1 \in K_\lambda^*$  which is universal over  $M_0$  for  $\leq_{\mathfrak{K}_\lambda}$ -extensions.

3) If (a) then (b) where

$$(a) \quad I_0 \leq_{K_\lambda^{\text{flin}}} I'_1 <_{K_\lambda^{\text{flin}}} I_1$$

$$(b) \quad \text{if } I_0 \subseteq I_2 \in K_\lambda^{\text{flin}} \text{ and } \beta \leq \gamma < \lambda, \bar{b}_1 \in {}^\beta(\text{EM}_{\tau(\mathfrak{K})}(I'_1, \Phi)) \text{ and } \bar{c}_2 \in {}^\gamma(\text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)) \text{ and } \bar{b}_2 = \bar{c}_2 \upharpoonright \beta \text{ and for every } \kappa < \lambda \text{ we have}$$

$$\begin{aligned} \text{tp}_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]}(\bar{b}_1, \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi), \text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi)) &= \\ &= \text{tp}_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]}(\bar{b}_2, \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi), \text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)) \end{aligned}$$

then for some  $(I_1^+, f)$  we have  $I_1 \leq_{K_\lambda^{\text{flin}}} I_1^+ \in K_\lambda^{\text{flin}}$  and  $f$  is a  $\leq_{\mathfrak{K}}$ -embedding of  $\text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$  into  $\text{EM}_{\tau(\mathfrak{K})}(I_1^+, \Phi)$  over  $\text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$  mapping  $\bar{b}_2$  to  $\bar{b}_1$  and  $\bar{c}_2$  into  $\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi)$ .

4) Assume  $\text{cf}(\lambda) = \aleph_0$ . If (c) then (d) and moreover (d)<sup>+</sup> when

$$(c) \quad \langle J_\alpha : \alpha \leq \omega \rangle \text{ is } <_{K_\lambda^{\text{flin}}}\text{-increasing, } I_0 = J_0, I_1 = J_\omega$$

$$(d) \quad \text{if } I_0 \subseteq I_2 \in K_\lambda^{\text{flin}} \text{ then some } f \text{ is a } \leq_{\mathfrak{K}}\text{-embedding of } \text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi) \text{ into } \text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \text{ over } \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$$

$$(d)^+ \quad \text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \text{ is } \leq_{\mathfrak{K}_\lambda^*}\text{-universal over } \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi).$$

*Proof.* Note that by 1.29(3) clearly (3)  $\Rightarrow$  (1) and (4)  $\Rightarrow$  (2). So we shall prove (3) and (4).

3) First assume  $\beta = 0, \gamma = 1$  so  $\bar{c}_2 = \langle c \rangle$ . Toward contradiction assume  $I_0 \subseteq I_2 \in K_\lambda^{\text{flin}}, a \in M_2 := \text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$  but there is no pair  $(I_1^+, f)$  as required in clause (b). Without loss of generality for some  $I_3$  we have  $I_0 \leq_{K_\lambda^{\text{flin}}} I_2 \leq_{K_\lambda^{\text{flin}}} I_3$  and  $I_0 \leq_{K_\lambda^{\text{flin}}} I_1 \leq_{K_\lambda^{\text{flin}}} I_3$ .

Let  $\text{EM}(I_2, \Phi) \models "c_2 = \sigma(a_{t_0^2}, \dots, a_{t_{n-1}^2})"$  where  $\sigma(x_0, \dots, x_{n-1})$  a  $\tau_\Phi$ -term,  $n < \omega$  and  $I_2 \models "t_0^2 < \dots < t_{n-1}^2"$ . Let  $u = \{\ell < n : t_\ell^2 \in I_0\}$ . As  $I_0 <_{K_\lambda^{\text{flin}}} I_1$ , we can find  $\langle t_0^1, \dots, t_{n_1}^1 \rangle$  such that:

- ⊗ (a)  $t_\ell^1 \in I_1$  for  $\ell < n$
- (b)  $t_0^1 <_{I_1} \dots <_{I_1} t_{n-1}^1$
- (c) if  $\ell \in u$  then  $t_\ell^2 = t_\ell^1 (\in I_0)$
- (d) if  $\ell < n \wedge \ell \notin u$  then  $t_\ell^1 \in I_1 \setminus I_0$
- (e) if  $\ell_1 \leq \ell_2 < n$  and  $[\ell_1, \ell_2] \cap u = \emptyset$  then  $t_{\ell_2}^2 <_{I_3} t_{\ell_1}^1$ .

Let  $M_\ell = \text{EM}_{\tau(\mathfrak{K})}(I_\ell, \Phi)$  for  $\ell = 0, 1, 2, 3$  and let  $c_2 = c$  and  $c_1 = \sigma^{\text{EM}(I_1, \Phi)}(a_{t_0^1}, \dots, a_{t_{n-1}^1})$ .

Let  $\kappa < \lambda$  be large enough such that  $\text{tp}_{\mathbb{L}_{\infty, \kappa+[\mathfrak{K}]}}(c_\ell, M_0, M_\ell)$  for  $\ell = 1, 2$  be distinct (exists by 1.29(1) because its conclusion fails by the “toward contradiction”). We easily get contradiction to the non-order property (see  $(*)$  of 1.5(2)).

Note that if in addition  $\langle I_{1,\alpha} : \alpha \leq \lambda \rangle$  is  $<_{K_\lambda^{\text{fin}}}$ -increasing continuous,  $I_{1,0} = I'_1, I_{1,\lambda} = I_1$  then by what we have just proved and the proof of III. 600-4a.2 we can prove the general case (and part (4)). But we also give a direct proof.

In the general case, let  $\theta = |\beta| + \aleph_0$ , so we assume clause (a) and the assumptions of clause (b) and without loss of generality  $I_1 \cap I_2 = I_0$  hence there is  $I_3$  such that  $I_\ell <_{K_\lambda^{\text{fin}}} I_3$  for  $\ell = 1, 2$ . Let  $\kappa \in (\theta, \lambda)$  be large enough.

Hence  $\text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi) \prec_{\mathbb{L}_{\infty, \lambda}[\mathfrak{K}]} \text{EM}_{\tau(\mathfrak{K})}(I_\ell, \Phi) \prec_{\mathbb{L}_{\infty, \lambda}[\mathfrak{K}]} \text{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$  for  $\ell = 1, 2$ . Applying 1.29(1) with  $I_1, I_2, \bar{b}, \bar{c}$  there standing for  $I_0, I_3, \bar{b}_1, \bar{b}_2$  here we can find a pair  $(I_4, f_4)$  such that  $I_3 <_{K_\lambda^{\text{fin}}} I_4$  and  $f_4$  is an automorphism of  $M_4 := \text{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$  over  $\text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$  mapping  $\bar{b}_2$  to  $\bar{b}_1$ .

Clearly  $M_3 := \text{EM}_{\tau(\mathfrak{K})}(I_3, \Phi) \prec_{\mathbb{L}_{\infty, \lambda}[\mathfrak{K}]} \text{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$ . So  $f_4(\bar{c}_2) \in \gamma(M_4)$ , hence we can apply clause (b) of Claim 1.26(3) with  $M_1, M_2, I_2, N, \xi, \bar{d}^*$  there standing for  $\text{EM}_{\tau(\mathfrak{K})}(I'_1, \Phi), \text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi), I_1,$

$\text{EM}_{\tau(\mathfrak{K})}(I_4, \Phi), \gamma, f_4(\bar{c}_2)$  here. Hence we can find  $\bar{c}'_2 \in \gamma(M_1)$  realizing in  $M_1$  the type  $\text{tp}_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]}(f_4(\bar{c}_2), \text{EM}_{\tau(\mathfrak{K})}(I'_1, \Phi), \text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi))$ .

Lastly, applying Claim 1.29(1) with  $I_1, I_2, \bar{b}, \bar{c}$  there standing for  $I'_1, I_4, f_4(\bar{c}_2), \bar{c}'_2$  here, clearly there is a pair  $(I_5, f_5)$  such that  $I_4 <_{K_\lambda^{\text{fin}}} I_5$  and  $f_5$  is an automorphism of  $\text{EM}_{\tau(\mathfrak{K})}(I_5, \Phi)$  over  $\text{EM}_{\tau(\mathfrak{K})}(I'_1, \Phi)$  mapping  $f_4(\bar{c}_2)$  to  $\bar{c}'_2$ .

Let  $I_1^+ := I_5, f = f'_5 \circ f'_4$  where  $f'_5 = f_5 \upharpoonright \text{EM}_{\tau(\mathfrak{K})}(I_4, \Phi), f'_4 = f_4 \upharpoonright \text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$ ; now  $I_1^+, f$  are as required because  $f_4(\bar{b}_2) = \bar{b}_1$  while  $f_5(\bar{b}_1) = \bar{b}_1$ .

4) Easy by part (3). First note that  $(d)^+$  follows by (d) by 1.29(3), so we shall ignore clause  $(d)^+$ . Let  $\text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$  be  $\cup\{M_{2,n} : n < \omega\}$  where  $M_{2,n} \in K_{<\lambda}$  and  $n < \omega \Rightarrow M_{2,n} \leq_{\mathfrak{K}} M_{2,n+1}$ .

Let  $\bar{a}_n$  list the elements of  $M_{2,n}$  with no repetitions such that  $\bar{a}_n \triangleleft \bar{a}_{n+1}$  for  $n < \omega$ . By induction on  $n$ , we choose  $\bar{b}_n$  such that


- ⊗ (a)  $\bar{b}_n \in \ell g(\bar{a}_n)(\text{EM}_{\tau(\mathfrak{K})}(J_{n+1}, \Phi)$
- (b) if  $n = m + 1$  then  $\bar{b}_m \triangleleft \bar{b}_n$
- (c) for every  $\kappa < \lambda$  the type  $\text{tp}_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]}(\bar{b}_n, \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi), \text{EM}_{\tau(\mathfrak{K})}(I_{n+1}, \Phi))$  is equal to the type  $\text{tp}_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]}(\bar{a}_n, \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi), \text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi))$ .

The induction step is by part (3). Let  $f_n$  be the unique function mapping  $\bar{a}_n$  to  $\bar{b}_n$  (with domain  $\text{Rang}(\bar{a}_n)$ ). So  $f_n \subseteq f_{n+1}$  and  $f_n$  is a  $\leq_{\mathfrak{K}}$ -embedding of  $M_{2,n}$  into  $\text{EM}_{\tau(\mathfrak{K})}(J_{n+1}, \Phi)$  but  $J_{n+1} \subseteq I_1$  hence into  $\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi)$ . So  $f := \cup\{f_n : n < \omega\}$  is a  $\leq_{\mathfrak{K}}$ -embedding of  $\text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$  into  $\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi)$ . Also  $f_n$  is the

identity on  $\text{Rang}(\bar{a}_n) \cap \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$  hence  $f$  is the identity on  $\bigcup_n (\text{Rang}(\bar{a}_n) \cap \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)) = \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$  so  $f$  is as required.  $\square_{1.32}$

**1.33 Exercise:** 1) Assume  $\mathfrak{K}_\lambda = (K_\lambda, \leq_{\mathfrak{K}_\lambda})$  satisfies axioms I,II (and 0, presented below) and amalgamation. Then  $\mathbf{tp}(a, M, N)$  for  $M \leq_{\mathfrak{K}_\lambda} N$  and  $a \in N$  and  $\mathcal{S}_{\mathfrak{K}_\lambda}(M)$  are well defined and has the basic properties of types from III§1.

2) If in addition  $\mathfrak{K}_\lambda$  satisfies  $\text{AxIII}^\odot$  below and  $\mathfrak{K}_\lambda$  is stable (i.e.  $|\mathcal{S}_{\mathfrak{K}_\lambda}(M)| \leq \lambda$  for  $M \in K_\lambda$ ) then every  $M \in \mathfrak{K}_\lambda$  has a  $\leq_{\mathfrak{K}}$ -universal extension  $N$  which means  $M \leq_{\mathfrak{K}_\lambda} N$  and  $(\forall N') (M \leq_{\mathfrak{K}_\lambda} N' \rightarrow (\exists f)[f \text{ is a } \leq_{\mathfrak{K}_\lambda}\text{-embedding of } N' \text{ into } N \text{ over } M])$ .

3)  $\text{AxIII}$  (see III.  600-0.2 ) implies  $\text{AxIII}^\odot$  where:

$\text{Ax0}$ :  $K$  is a class of  $\tau_{\mathfrak{K}}$ -models,  $\leq_{\mathfrak{K}}$  a two place relation of  $K_\lambda$ , both preserved under isomorphisms

$\text{AxI}$ : if  $M \leq_{\mathfrak{K}_\lambda} N$  then  $M \subseteq N$  (are  $\tau(\mathfrak{K}_\lambda)$ -models of cardinality  $\lambda$ )

$\text{AxII}$ :  $\leq_{\mathfrak{K}_\lambda}$  is a partial order (so  $M \leq_{\mathfrak{K}_\lambda} M$  for  $M \in K_\lambda$ )

$\text{AxIII}^\odot$ : In following game the COM player has a winning strategy. A play last  $\lambda$  moves, they construct a  $\leq_{\mathfrak{K}_\lambda}$ -increasing continuous sequence  $\langle M_\alpha : \alpha \leq \lambda \rangle$ . In the  $\alpha$ -th move  $M_\alpha$  is chosen, by INC if  $\alpha$  is even by COM if  $\alpha$  is odd. Now Com wins as long as INC has legal moves.

$\text{AxIV}^\odot$ : For each  $M \in K_\lambda$ , in the following game, INC has no winning strategy: a play lasts  $\lambda + 1$  moves, in the  $\alpha$ -th move  $f_\alpha, M_\alpha, N_\alpha$  are chosen such that  $f_\alpha$  is a  $\leq_{\mathfrak{K}}$ -embedding of  $M_\alpha$  into  $N_\alpha$ , both are  $\leq_{\mathfrak{K}_\lambda}$ -increasing continuous,  $f_\alpha$  is  $\subseteq$ -increasing continuous,  $M_0 = M$  and in the  $\alpha$ -th move,  $M_\alpha$  is chosen by INC, and the pair is chosen by the player INC if  $\alpha$  is even and by the player COM if  $\alpha$  is odd. The player COM wins if INC has always a legal move (the player COM always has: he can choose  $N_\alpha = M_\alpha$ )

**1.34 Definition.** 1) Let  $<^*_\lambda = <^*_{\mathfrak{K}_\lambda}$  be the following two-place relation on  $K_\lambda^*$  (so  $M \leq^*_{\mathfrak{K}_\lambda} N$  mean  $M = N \in \mathfrak{K}_\lambda^*$  or  $M <^*_{\mathfrak{K}_\lambda} N$ ):

$M_1 <^*_\lambda M_2$  iff  $M_1 \leq_{\mathfrak{K}_\lambda} M_2$  are from  $K_\lambda^*$  and  $M_2$  is  $\leq_{\mathfrak{K}_\lambda}$ -universal over  $M_1$ .

2) For  $\alpha < \lambda, \kappa = \beth_{1,1}(|\alpha| + \text{LS}(\mathfrak{K}))$  and  $M \in K_\lambda^*$  let  $\text{Sav}^{\text{bs}, \alpha}(M)$  be the set of  $\{\text{Av}_\kappa(\mathbf{I}, M) : \mathbf{I} \text{ is a } ((2^\kappa)^+, \kappa)\text{-convergent subset of } {}^\alpha M\}$ . We define  $\text{tp}_*(\bar{a}, M, N)$  when  $M \leq_{\mathfrak{K}} N$  are from  $K_\lambda^*$  and  $\bar{a} \in {}^\alpha N$ , as  $\text{tp}_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]}(\bar{a}, M, N) \in \text{Sav}^{\text{bs}, \alpha}(M)$

naturally.

3) Let  $\mathfrak{K}_\lambda^* = (K_\lambda^*, \leq_\mathfrak{K} \upharpoonright \mathfrak{K}_\lambda^*, \leq_{\mathfrak{K}_\lambda^*}^*)$ , see 1.35 below but if  $(K_\lambda^*, \leq_\mathfrak{K} \upharpoonright K_\lambda^*)$  is a  $\lambda$ -a.e.c. then we omit  $\leq_{\mathfrak{K}_\lambda^*}^*$ .

1.35 Remark. 1) Note that the relation  $<_\lambda^* = <_{\mathfrak{K}_\lambda^*}^*$  seemingly depends on the choice of  $\Phi$ . However, assuming  $\mu$ -solvability, by 1.37(2) below it does not depend.

2) The proof of 1.37 is like III. 600-0.22 (3).

3) So  $\mathfrak{K}_\lambda^*$  is a semi- $\lambda$ -a.e.c. (see Chapter N) but we do not use this notion here.

**1.36 Claim.** Assume  $\text{cf}(\lambda) = \aleph_0$ .

0) If  $M \in K_\lambda^*$  then for some  $N$ ,  $M <_{\mathfrak{K}_\lambda^*}^* N (\in K_\lambda^*)$ .

1) If  $M \leq_\mathfrak{K} N$  are from  $K_\lambda^*$ ,  $\alpha < \lambda$  and  $\bar{a} \in {}^\alpha N \setminus {}^\alpha M$  then  $\bar{a}$  realizes some  $p \in \text{Sav}^{\text{bs}, \alpha}(M)$ .

2) If  $M_0 \leq_\mathfrak{K} M_1 <_{\mathfrak{K}_\lambda^*}^* M_2 \leq_\mathfrak{K} M_3$  and  $M_\ell \in K_\lambda^*$  for  $\ell < 4$ , then  $M_0 <_{\mathfrak{K}_\lambda^*}^* M_3$ .

*Proof.* 0) As  $K_\lambda^*$  is categorical (by 1.16(1)) this follows by 1.32(2).

1) A proof of this is included in the proof of 1.29(2), i.e. by 1.26(1).

2) Easy recalling amalgamation. □<sub>1.36</sub>

**1.37 Claim.** Assume  $\text{cf}(\lambda) = \aleph_0$ .

1) Assume  $\langle M_i : i \leq \delta \rangle$  is  $\leq_{\mathfrak{K}_\lambda}$ -increasing continuous,  $M_{2i+1} <_{\mathfrak{K}_\lambda^*}^* M_{2i+2}$  for  $i < \delta$  then  $M_\delta \in K_\lambda^*$ .

2) Assume that  $\langle M_i^\ell : i \leq \delta \rangle$  is an  $\leq_{\mathfrak{K}_\lambda^*}$ -increasing continuous sequence such that  $M_{2i+1}^\ell <_{\mathfrak{K}_\lambda^*}^* M_{2i+2}^\ell$  for  $i < \delta$  all for  $\ell = 1, 2$ . Any isomorphism  $f$  from  $M_0^1$  onto  $M_0^2$  (or just a  $\leq_{\mathfrak{K}_\lambda}$ -embedding) can be extended to an isomorphism from  $M_\delta^1$  onto  $M_\delta^2$ .

*Proof.* 1) We prove this by induction on  $\delta$ , hence without loss of generality  $i < \delta \Rightarrow M_i \in K_\lambda^*$ .

Let  $M_\alpha^1 = M_\alpha$  for  $\alpha \leq \delta$  and let  $\langle I_\alpha : \alpha \leq \delta \rangle$  be  $<_{K_\lambda^{\text{flin}}}$ -increasing. Let  $M_\alpha^2 = \text{EM}_{\tau(\mathfrak{K})}(I_\alpha, \Phi)$ . Now there is an isomorphism  $f$  from  $M_0^1$  onto  $M_0^2$  as  $K_\lambda^*$  is categorical, so by part (2) there is an isomorphism  $g$  from  $M_\alpha^1$  onto  $M_\alpha^2$ , but  $M_\alpha^2 \in K_\lambda^*$  so we are done.

2) Note

□<sub>2</sub> without loss of generality

$$\square \quad M_i^2 <_\lambda^* M_{i+1}^2.$$

[Why? We can find  $\langle M_i^3 : i \leq \delta \rangle$  which is  $\leq_{\mathfrak{K}_\lambda^*}^*$ -increasing continuous and  $M_0^3 = M_0^2$  and  $M_i^3 <_\lambda^* M_{i+1}^3$ . Now apply the restricted version (i.e., with the assumption  $\square$ ) twice.]

By induction on  $i \leq \delta$  we choose  $(f_i, N_i^1, N_i^2)$  such that

- ⊗ (a)  $N_i^1, N_i^2$  belongs to  $K_\lambda^*$
- (b)  $f_i$  is an isomorphism from  $N_i^1$  onto  $N_i^2$
- (c)  $N_i^1, N_i^2, f_i$  are increasing continuous with  $i$
- (d) for  $i = 0$ ,  $N_i^1 = M_i^1$ ,  $f_i = f$  and  $N_i^2$  is  $f(M_i^1) = M_i^2$
- (e) if  $i > 0$  is a limit ordinal then  $N_i^1 = M_i^1$  and  $N_i^2 = M_i^2$
- (f) when  $i = \omega\alpha + 2n < \delta$  we have
  - (α)  $N_{\omega\alpha+2n+1}^1 = M_{\omega\alpha+2n+1}^1$
  - (β)  $N_{\omega\alpha+2n+1}^2 \leq_{\mathfrak{K}} M_{\omega\alpha+2n+1}^2$
  - (γ)  $N_{\omega\alpha+2n+2}^1 \leq_{\mathfrak{K}} M_{\omega\alpha+2n+2}^1$
  - (δ)  $N_{\omega\alpha+2n+2}^2 = M_{\omega\alpha+2n+2}^2$ .

Case 1: For  $i = 0$  this is trivial by clause (d) and the assumption of the claim on  $f$ .

Case 2:  $i = \omega\alpha + 2n + 1$ .

Note that  $N_{\omega\alpha+2n}^2 = M_{\omega\alpha+2n}^2$ . (Why? If  $i = 0$  (i.e.  $\alpha = 0 = n$ ) by ⊗(d) and if  $i$  is a limit ordinal (i.e.  $\alpha > 0 \wedge n = 0$ ) by clause (e) of ⊗ and if  $n > 0$  by clause ((f)(δ) of ⊗).

Now we let  $N_i^1 = N_{\omega\alpha+2n+1}^1 := M_{\omega\alpha+2n+1}^1$  and hence satisfying clause (f)(α) of ⊗. So  $N_{i-1}^1 = N_{\omega\alpha+2n}^1 \leq_{\mathfrak{K}} M_{\omega\alpha+2n}^1 \leq_{\mathfrak{K}} M_{\omega\alpha+2n+1}^1 = N_{\omega\alpha+2n+1}^1 = N_i^1$ ; and note that  $N_{i-1}^2 = N_{\omega\alpha+2n}^2 <_\lambda^* M_{\omega\alpha+2n}^2$  by  $\square$  above hence we can apply Definition 1.34(1) and find an extension  $f_i$  of  $f_{i-1}$  to  $\leq_{\mathfrak{K}}$ -embedding of  $N_i^1 = M_{\omega\alpha+2n+1}^1$  into  $M_{\omega\alpha+2n+1}^2$  and let  $N_i^2 := f_i(N_i^1)$ .

Case 3:  $i = \omega\alpha + 2n + 2$ .

Note that  $N_{\omega\alpha+2n+1}^1 = M_{\omega\alpha+2n+1}^1$  by clause (f)(α) of ⊗ hence by the assumption of the claim  $N_{\omega\alpha+2n+1}^1 <_{\mathfrak{K}_\lambda^*}^* M_{\omega\alpha+2n+2}^1$ . We choose  $N_{\omega\alpha+2n+2}^2 := M_{\omega\alpha+2n+2}^2$  hence  $N_{i-1}^2 = N_{\omega\alpha+2n+1}^2 \leq_{\mathfrak{K}} M_{\omega\alpha+2n+1}^2 \leq_{\mathfrak{K}} M_{\omega\alpha+2n+2}^2 = N_{\omega\alpha+2n+2}^2 = N_i^2$ . Now we apply Definition 1.34(1) to find a  $\leq_{\mathfrak{K}}$ -embedding  $g_i$  of  $N_{\omega\alpha+2n+2}^2$  into  $M_{\omega\alpha+2n+2}^1$  extending  $f_{i-1}^{-1}$ .

Lastly, let  $f_i = g_i^{-1}$  and  $N_i^1 = M_i^1 \upharpoonright \text{Dom}(f_i)$ . So we can carry the induction hence prove the claim.  $\square_{1.37}$

Note that now we use more than in Hypothesis 1.18.

**1.38 Claim.** *Assume*

- ⊠ (a)  $\langle \lambda_n : n < \omega \rangle$  is increasing,  $\lambda = \lambda_\omega = \sum_{n < \omega} \lambda_n$  satisfying  
 $\lambda_n = \beth_{\lambda_n} > \text{LS}(\mathfrak{K})$  and  $\text{cf}(\lambda_n) = \aleph_0$  for  $n < \omega$   
 (b)  $\Phi \in \Upsilon_{\mathfrak{K}}^{\text{or}}$  and for it each  $\lambda_n$  and  $\lambda = \lambda_\omega$  is as in Hypothesis 1.18  
 or just satisfies all its conclusions so far.

- 1)  $K_\lambda^*$  is closed under unions  $\leq_{\mathfrak{K}}$ -increasing chains (of length  $< \lambda^+$ ).  
 2) If  $M_n \in K_{\lambda_n}^*$ ,  $M_n \leq_{\mathfrak{K}} M_{n+1}$  and  $M = \bigcup_{n < \omega} M_n$  then  $M \in K_\lambda^*$ .  
 3) If  $M \in K_\lambda$  and  $\theta < \lambda \Rightarrow M \equiv_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]} \text{EM}_{\tau(\mathfrak{K})}(\lambda, \Phi)$  then  $M \in K_\lambda^*$ .  
 4)  $K_\lambda^*$  is categorical.

*Proof of 1.38.* 1) We rely on part (2) which is proven below.

So let  $\langle M_i : i < \delta \rangle$  be  $\leq_{\mathfrak{K}}$ -increasing in  $K_\lambda^*$  with  $\delta < \lambda^+$ . Without loss of generality  $\delta = \text{cf}(\delta)$  hence  $\delta < \lambda$  so call it  $\theta$  and we prove this by induction on  $\theta$ , so without loss of generality  $\langle M_i : i < \theta \rangle$  is  $\leq_{\mathfrak{K}}$ -increasing continuous such that  $M_i \in K_\lambda^*$  for  $i < \theta$ , and let  $M_\theta = \bigcup_{i < \theta} M_i$ . By renaming without loss of generality  $\theta < \lambda_0$ .

Let  $I_n, I'_n$  be such that:

- ⊙<sub>1</sub> (a)  $I_n$  is a linear order of cardinality  $\lambda_n$  from  $K^{\text{flin}}$   
 (b)  $I'_n$  is a linear order of cardinality  $2^{\lambda_n}$  from  $K^{\text{flin}}$   
 (c)  $I'_n$  is  $\lambda_n^+$ -saturated (which means that its cofinality is  $> \lambda_n$ , the cofinality of its inverse is  $> \lambda_n$  and if  $I'_n \models "s_{\alpha_1} < s_{\beta_1} < t_{\beta_2} < t_{\alpha_2}"$  where  $\alpha_1 < \beta_1 < \gamma_1, \alpha_1 < \beta_2 < \gamma_2$  and  $|\gamma_1| + |\gamma_2| < \lambda_n^+$  then for some  $r$  we have  $I'_n \models "s_{\alpha_1} < r < t_{\alpha_2}"$  for  $\alpha_1 < \gamma_1, \alpha_2 < \gamma_2$ )  
 (d)  $I_n <_{K^{\text{flin}}} I'_n <_{K^{\text{flin}}} I_{n+1}$  for  $n < \omega$ .

Let  $I = \bigcup \{I_n : n < \omega\}$ , so  $I$  is a universal member of  $K_\lambda^{\text{lin}}$ . Let  $M^* = \text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ , so for every  $i < \theta$  there is an isomorphism  $f_i$  from  $M^*$  onto  $M_i$ , exists as  $K_\lambda^*$  is categorical by 1.19(4) as  $\text{cf}(\lambda) = \aleph_0$ .

Now

- ⊙<sub>2</sub> (a) every interval of  $I$  is universal in  $K_\lambda^{\text{lin}}$   
 (b) if  $n < \omega, J \subseteq I, \chi = |J| < \lambda$  and  $\mathcal{E}_{J,I} = \{(t_1, t_2) : t_1, t_2 \in I \setminus J \text{ and } s \in J \Rightarrow s <_I t_1 \equiv s <_J t_2\}$  then for at most  $\chi$  elements of  $t$  of  $J \setminus I$  the set  $t/\mathcal{E}_{J,I}$  is a singleton.



[Why? Clause (a) is obvious. For clause (b) assume  $\langle t_\alpha : \alpha < \chi^+ \rangle$  are pairwise distinct members of  $J \setminus I$  such that  $t_\alpha / \mathcal{E}_{J,I}$  is a singleton for each  $\alpha < \chi^+$ . Without loss of generality for some  $k < \omega$  we have  $\alpha < \chi^+ \Rightarrow t_\alpha \in I_k$  hence  $\chi \leq \lambda_k$ . For each  $\alpha < \chi^+$  we can choose  $s_\alpha \in I'_k$  such that  $s_\alpha <_{I'_k} t_\alpha$  and  $(s_\alpha, t_\alpha)_{I'_k} \cap J = \emptyset$ . Clearly  $\alpha < \beta < \chi^+ \Rightarrow (t_\alpha <_I s_\beta \vee t_\beta <_I s_\alpha)$  hence  $\langle (s_\alpha, t_\alpha)_I : \alpha < \chi^+ \rangle$  are pairwise disjoint intervals of  $I$ , so for every  $\alpha < \chi^+$  large enough,  $(s_\alpha, t_\alpha)_I \cap J = \emptyset$ , but then  $(s_\alpha, t_\alpha)_I \subseteq t_\alpha / \mathcal{E}_{J,I}$ , contradiction.]

Now by induction on  $n < \omega$  and for each  $n$  by induction on  $\varepsilon \leq \theta$  and for each  $n < \omega$  and  $\varepsilon \leq \theta$  for  $i \leq \theta$ , we choose  $J_{n,\varepsilon,i} \in K_{\lambda_n}^{\text{fin}}$  such that:

- ⊙<sub>3</sub> (a)  $J_{n,\varepsilon,i} \subseteq I$
- (b)  $J_{n,\varepsilon,i}$  has cardinality  $\lambda_n$
- (c)  $I_n <_{K^{\text{fin}}} J_{n,0,i}$
- (d) if  $\zeta < \varepsilon \leq \theta$  and  $i \leq \theta$  then  $J_{n,\zeta,i} \subseteq J_{n,\varepsilon,i}$ , moreover if for some  $\xi, \zeta = 2\xi + 1$  and  $\varepsilon = 2\xi + 2$  then there is a  $<_{K_{\lambda_n}^{\text{fin}}}$ -increasing continuous sequence of length  $\omega$  with first member  $J_{n,\zeta,i}$  and union  $J_{n,\varepsilon,i}$
- (e) for  $\varepsilon$  limit,  $J_{n,\varepsilon,i} = \bigcup_{\zeta < \varepsilon} J_{n,\zeta,i}$
- (f) if  $\varepsilon$  is odd and  $i < j < \theta$  then  $f_i(\text{EM}_{\tau(\mathfrak{K})}(J_{n,\varepsilon,i}, \Phi)) = M_i \cap f_j(\text{EM}_{\tau(\mathfrak{K})}(J_{n,\varepsilon,j}, \Phi))$
- (g)  $J_{n,\theta,i} \subseteq J_{n+1,0,i}$
- (h) for every  $k < \omega$  and  $s <_I t$  from  $J_{n,\varepsilon,i}$  if  $[s, t]_I \cap I'_k \neq \emptyset$  then  $[s, t]_I \cap I'_k \cap J_{n,\varepsilon,i} \neq \emptyset$
- (i) if  $\zeta$  is odd and  $\varepsilon = \zeta + 1$  then  $\text{EM}_{\tau(\mathfrak{K})}(J_{n,\zeta,i}, \Phi) <_{\mathfrak{K}_{\lambda_n}^*}^* \text{EM}_{\tau(\mathfrak{K})}(J_{n,\varepsilon,i}, \Phi)$ .

There is no problem to carry the definition, for  $\varepsilon = 2\xi + 2$  recalling ⊙<sub>2</sub> above; the only non-trivial point is clause (i), which follows by 1.32(4) and clause (d) of ⊙<sub>3</sub>. Clearly  $\langle J_{n,\varepsilon,i} : \varepsilon \leq \theta \rangle$  is  $\subseteq$ -increasing continuous by ⊙<sub>3</sub>(d) + (e).

Let  $M_{n,\varepsilon,i}^* = f_i(\text{EM}_{\tau(\mathfrak{K})}(J_{n,\varepsilon,i}, \Phi))$  and  $M_{n,\varepsilon}^* = M_{n,2\varepsilon,\varepsilon}^*$ . So clearly  $M_{n,\varepsilon,i}^* \in K_{\lambda_n}^*$  by ⊙<sub>3</sub>(b) and the choice of  $M_{n,\varepsilon,i}^*$  the sequence  $\langle M_{n,\varepsilon}^* : \varepsilon < \theta \rangle$  is  $\leq_{\mathfrak{K}}^*$ -increasing continuous, all members in  $K_{\lambda_n}^*$ .

Now

- ⊙<sub>4</sub>  $\langle M_{n,\varepsilon}^* : \varepsilon < \theta \rangle$  is  $<_{\mathfrak{K}_{\lambda_n}^*}^*$ -increasing.

[Why? As  $\zeta < \varepsilon < \theta \Rightarrow M_{n,\zeta}^* = M_{n,2\zeta,\zeta} \leq_{\mathfrak{K}_{\lambda_n}^*} M_{n,2\zeta+1,\zeta} \leq_{\mathfrak{K}_{\lambda_n}^*} M_{n,2\zeta+1,\varepsilon} <_{\mathfrak{K}_{\lambda_n}^*}^* M_{n,2\zeta+2,\varepsilon} \leq_{\mathfrak{K}_{\lambda_n}^*} M_{n,2\varepsilon,\varepsilon} = M_{n,\varepsilon}^*$  by the choice of  $M_{n,\zeta}^*$ , by ⊙<sub>3</sub>(d) and Ax(V) of a.e.c., by ⊙<sub>3</sub>(f) and Ax(V) of a.e.c., by ⊙<sub>3</sub>(i), by ⊙<sub>3</sub>(d) + Ax(V) of a.e.c.(e),

by the choice of  $M_{n,\varepsilon}^*$  respectively). Now by 1.36(2) this argument shows that  $\zeta < \varepsilon < \theta \Rightarrow M_{n,\zeta}^* <_{\mathfrak{K}_{\lambda_n}^*}^* M_{n,\varepsilon}^*$

We can conclude by using 1.37(1) for  $\mathfrak{K}_{\lambda_n}^*$ , that  $M_n^* := \bigcup_{\varepsilon < \theta} M_{n,\varepsilon}^*$  belongs to  $K_{\lambda_n}^*$ . Also as  $M_{n,\varepsilon}^* \leq_{\mathfrak{K}} M_\varepsilon \leq_{\mathfrak{K}} M_\delta$  for  $\varepsilon < \theta = \delta$  by AxIV of a.e.c. we have  $M_n^* \leq_{\mathfrak{K}} M_\delta$  and similarly  $M_n^* \leq_{\mathfrak{K}} M_{n+1}^*$ , and obviously for each  $i < \theta$  we have  $\bigcup_{n < \omega} M_n^*$  includes  $\cup\{M_{n,\varepsilon}^* : n < \omega, \varepsilon < \theta\} = \cup\{M_{n,2,\varepsilon}^* : n < \omega, \varepsilon < \theta\} = \cup\{M_{n,2\varepsilon,i}^* : n < \omega, i < \theta, \varepsilon < \theta\} = \bigcup_{n < \omega} M_{n,0,i}^*$  which recalling the choice of  $M_{n,0,i}^*$  includes  $\bigcup_n f_i(\text{EM}_{\tau(\mathfrak{K})}(J_{n,0,i}, \Phi)) \supseteq \bigcup_{n < \omega} f_i(\text{EM}_{\tau(\mathfrak{K})}(I_n, \Phi)) = f_i(\text{EM}_{\tau(\mathfrak{K})}(I, \Phi)) = M_i$ . As this holds for every  $i < \theta$  we get  $\bigcup_{n < \omega} M_n^* = M_\delta$ . So by part (2) we are done.

2) We choose  $I_n$  by induction on  $n$  such that:

- $\odot_5$  (a)  $I_n \in K_{\lambda_n}^{\text{flin}}$
- (b)  $I_m <_{K^{\text{flin}}} I_n$  if  $n = m + 1$ .

Let  $N_n = \text{EM}_{\tau(\mathfrak{K})}(I_n, \Phi)$ .

We now choose  $(g_n, I'_n, I''_n, M'_n, M''_n, N'_n, N''_n)$  by induction on  $n < \omega$  such that:

- $\odot_6$  (a)  $g_n$  is an isomorphism from  $N''_n$  onto  $M''_n$
- (b)  $I_n \subseteq I'_n \subseteq I''_n \subseteq I_{n+2}$  and  $|I'_n| = \lambda_n, |I''_n| = \lambda_{n+1}$  and  $I_{n+1} \subseteq I''_n$
- (c)  $N'_n = \text{EM}_{\tau(\mathfrak{K})}(I'_n, \Phi)$  and  $N''_n = \text{EM}_{\tau(\mathfrak{K})}(I''_n, \Phi)$
- (d)  $M_n \leq_{\mathfrak{K}_{\lambda_n}^*} M'_n \leq_{\mathfrak{K}^*} M''_n \leq_{\mathfrak{K}^*} M_{n+2}$  and  $M_{n+1} \leq_{\mathfrak{K}_{\lambda_{n+1}}^*} M''_n$
- (e)  $g_n$  maps  $N'_n = \text{EM}_{\tau(\mathfrak{K})}(I'_n, \Phi)$  onto  $M'_n$
- (f)  $g_n$  extends  $g_m \upharpoonright N'_m$  if  $n = m + 1$
- (g)  $I'_n \subseteq I'_{n+1}$ .

Case 1: For  $n = 0$ .

First, let  $M''_0 = M_1, I''_0 = I_1$  so also  $N''_0$  is defined. Second, choose  $g_n$  satisfying (a) of  $\odot_6$  by 1.16(1), i.e. 1.19(4), categoricity in  $K_{\lambda_n}^*$ . Third, choose  $I_n^* \subseteq I''_0 = I_1$  of cardinality  $\lambda_n$  such that  $g_n(\text{EM}_{\tau(\mathfrak{K})}(I_n^*, \Phi))$  includes  $M_0$ . Fourth, let  $I'_n = I_n^* \cup I_n$  and  $N'_n = \text{EM}_{\tau(\mathfrak{K})}(I'_n, \Phi)$  and let  $M'_n = g_n(N'_n)$ .

Case 2: For  $n = m + 1$ .

Let  $k = n + 2$ , let  $\bar{a} \in {}^{\lambda_m}(M'_m)$  list  $M'_m$  (with no repetitions).

Now

- $(*)_1$  If  $\theta < \lambda_n$  then  $\text{tp}_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]}(\bar{a}, \emptyset, N_k) = \text{tp}_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]}(\bar{a}, \emptyset, N''_m)$ .

[Why? As  $\text{EM}_{\tau(\mathfrak{K})}(I''_m, \Phi) \prec_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]} \text{EM}_{\tau(\mathfrak{K})}(I_k, \Phi)$  by 1.14(a) as  $I''_m \subseteq I_k$ .]

(\*)<sub>2</sub> if  $\theta < \lambda_n = \lambda_{m+1}$  then  
 $\text{tp}_{\mathbb{L}_{\infty, \theta}}(\bar{a}, \emptyset, N''_m) = \text{tp}_{\mathbb{L}_{\infty, \theta}}(g_m(\bar{a}), \emptyset, M''_m).$

[Why? As  $g_m$  is an isomorphism from  $N''_m$  onto  $M''_m$  by  $\odot_6(a)$ , i.e. the induction hypothesis.]

(\*)<sub>3</sub> if  $\theta < \lambda_n$  then  $\text{tp}_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]}(g_m(\bar{a}), \emptyset, M''_m) = \text{tp}_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]}(g_m(\bar{a}), \emptyset, M_k).$

[Why? This follows from  $M'_m \prec_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]} M_k$  which we can deduce by 1.19(1) as  $M''_m \in K_{\lambda_{m+1}}^* = K_{\lambda_n}^*$  by clause (d) of  $\odot_6$ ,  $M_k \in K_k^*$  by an assumption of the claim,  $M''_m \leq_{\mathfrak{K}_\lambda} M_k$  by clause (d) of  $\odot_6$ .]

(\*)<sub>4</sub> if  $\theta < \lambda_n$  then  $\text{tp}_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]}(\bar{a}, \emptyset, N_k) = \text{tp}_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]}(g_m(\bar{a}), \emptyset, M_k).$

[Why? By (\*)<sub>1</sub> + (\*)<sub>2</sub> + (\*)<sub>3</sub>.]

(\*)<sub>5</sub>  $\text{tp}_{\mathbb{L}_{\infty, \lambda_{n+1}^+}[\mathfrak{K}]}(\bar{a}, \emptyset, N_k) = \text{tp}_{\mathbb{L}_{\infty, \lambda_{n+1}^+}[\mathfrak{K}]}(g_m(\bar{a}), \emptyset, M_k).$

[Why? Clearly  $N_k, M_k \in K_{\lambda_k}^*$  hence by 1.19(4) there is an isomorphism  $f_n$  from  $N_k$  onto  $M_k$ , so obviously  $\text{tp}_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]}(\bar{a}, \emptyset, N_k) = \text{tp}_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]}(f_n(\bar{a}), \emptyset, N_k)$  so by (\*)<sub>4</sub> we have  $\text{tp}_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]}(g_m(\bar{a}), \emptyset, M_k) = \text{tp}_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]}(\bar{a}, \emptyset, N_k) = \text{tp}_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]}(f_n(\bar{a}), \emptyset, M_k)$  so by 1.19(3) we have  $\text{tp}_{\mathbb{L}_{\infty, \lambda_{n+1}^+}[\mathfrak{K}]}(g_n(\bar{a}), \emptyset, M_k) = \text{tp}_{\mathbb{L}_{\infty, \lambda_{n+1}^+}[\mathfrak{K}]}(f_n(\bar{a}), \emptyset, M_k)$ . But as  $f_n$  is an isomorphism from  $N_k$  onto  $M_k$  and the previous sentence we get  $\text{tp}_{\mathbb{L}_{\infty, \lambda_{n+1}^+}[\mathfrak{K}]}(\bar{a}, \emptyset, N_k) = \text{tp}_{\mathbb{L}_{\infty, \lambda_{n+1}^+}[\mathfrak{K}]}(f_n(\bar{a}), \emptyset, M_k) = \text{tp}_{\mathbb{L}_{\infty, \lambda}[\mathfrak{K}]}(g_n(\bar{a}), \emptyset, M_k)$  as required.]

(\*)<sub>6</sub> there are  $g_n, I''_n, N''_n, M''_n$  as required in the relevant parts of  $\odot_6$  (ignoring  $I'_n, N'_n, M'_n$ ), i.e. clauses (a),(f) and the relevant parts of (b),(c),(d):

(b)'  $I_n \subseteq I''_n \subseteq I_{n+2} = I_k$  and  $|I''_n| = \lambda_{n+1}$  and  $I_{n+1} \subseteq I'_n$

(c)'  $N''_n = \text{EM}_{\tau(\mathfrak{K})}(I''_n, \Phi)$

(d)'  $M_n \leq_{\mathfrak{K}^*} M''_n \leq_{\mathfrak{K}^*} M_{n+2}$  and  $M_{n+1} \leq_{\mathfrak{K}_{\lambda_{n+2}}^*} M''_n$ .

[Why? By the hence and forth argument, but let us elaborate.

First, let  $\bar{a}'$  be a sequence of length  $\lambda_{n+1}$  listing (without repetitions) the set of elements of  $M_{n+1}$  and without loss of generality  $g(\bar{a}) \triangleleft \bar{a}'$ . Note that  $\text{Rang}(g_m) \subseteq M_{m+2} = M_{n+1}$ .

Second, let  $g'$  be a function from  $\text{Rang}(\bar{a}')$  into  $N_k$  extending  $(g_m \upharpoonright N'_m)^{-1} = (g_m \upharpoonright \text{Rang}(\bar{a}))^{-1}$  such that  $\text{tp}_{\mathbb{L}_{\infty, \lambda_{n+1}^+}[\mathfrak{K}]}(g'(\bar{a}'), \emptyset, N_k) = \text{tp}_{\mathbb{L}_{\infty, \lambda_{n+1}^+}[\mathfrak{K}]}(\bar{a}', \emptyset, M_k),$

exists by  $(*)_5$ . Let  $I''_n \subseteq I_k$  of cardinality  $\lambda_{n+1}$  be such that  $\text{Rang}(g') \subseteq \text{EM}(I''_n, \Phi)$  and  $I_{n+1} \subseteq I''_n$ . Let  $\bar{a}''$  list the elements of  $\text{EM}_{\tau(\mathfrak{K})}(I''_n, \Phi) \subseteq N_k$  and without loss of generality  $g'(\bar{a}') \triangleleft \bar{a}''$  and let  $g_n$  be a function from  $\text{EM}_{\tau(\mathfrak{K})}(I''_n, \Phi)$  to  $M_k$  extending  $(g')^{-1}$  such that  $\text{tp}_{\mathbb{L}_{\infty, \lambda_{n+1}^+}[\mathfrak{K}]}(\bar{a}'', \emptyset, N_k) = \text{tp}_{\mathbb{L}_{\infty, \lambda_{n+1}^+}[\mathfrak{K}]}(g_n(\bar{a}''), \emptyset, M_k)$ .

Lastly, let  $N''_n = \text{EM}_{\tau(\mathfrak{K})}(I''_n, \Phi)$  and  $M''_n = g_n(N''_n)$  so we are done.]

$(*)_7$  there are  $I'_n, N'_n, M'_n$  as required.



[Why? By the LS argument we can choose  $I'_n$  and define  $N'_n, M'_n$  accordingly.]





So we can carry the induction. Now  $N'_n \leq_{\mathfrak{K}} N'_{n+1}$  (by clauses (g),(c) of  $\odot_6$ ) and  $g_n \upharpoonright N'_n \subseteq g_{n+1} \upharpoonright N'_{n+1}$  (by clause (f) + the previous statement). Hence  $g = \cup\{g_n \upharpoonright N'_n : n < \omega\}$  is an isomorphism from  $\cup\{N'_n : n < \omega\}$  onto  $\cup\{M'_n : n < \omega\}$ . But  $N = \cup\{N_n : n < \omega\} \subseteq \cup\{N'_n : n < \omega\} \subseteq \text{Dom}(g) \subseteq N$  and  $M = \cup\{M_n : n < \omega\} \subseteq \cup\{M'_n : n < \omega\} \subseteq \text{Rang}(g) \subseteq M$ . Together  $g$  is an isomorphism from  $N$  onto  $M$  but obviously  $N \in K_\lambda^*$  hence  $M \in K_\lambda^*$  is as required. 3),4) Should be clear and depends just on 1.19(4).  $\square_{1.38}$

**1.39 Conclusion.** Let  $\lambda$  be as in  $\boxtimes$  of 1.38.

- 1)  $\mathfrak{K}_\lambda^*$  is a  $\lambda$ -a.e.c. (with  $\leq_{\mathfrak{K}} \upharpoonright K_\lambda^*$ ) and it has amalgamation and is categorical.
- 2)  $\mathfrak{K}_{\geq \lambda}^\oplus$  is an a.e.c.,  $\text{LS}(\mathfrak{K}_{\geq \lambda}^\oplus) = \lambda$  and  $(\mathfrak{K}_\lambda^*)^{\text{up}} = K_{\geq \lambda}^\oplus$  and  $(\mathfrak{K}_{\geq \lambda}^\oplus)_\lambda = \mathfrak{K}_\lambda^*$ , see Definition below.

**1.40 Definition.** Let  $\mathfrak{K}_{\geq \lambda}^\oplus = \mathfrak{K} \upharpoonright K_{\geq \lambda}^\oplus$  where  $K_{\geq \lambda}^\oplus = \{M \in K_\lambda : M \equiv_{\mathbb{L}_{\infty, \lambda}[\mathfrak{K}]} \text{EM}_{\tau(\mathfrak{K})}(\lambda, \Phi)\}$ .

*Proof.* 1) It was clear defining  $(K_\lambda^*, \leq_{\mathfrak{K}} \upharpoonright K_\lambda^*)$  that it is of the right form and “ $M \in K_\lambda^*$ ”, “ $M \leq_{\mathfrak{K}} N$ ” are preserved by isomorphisms. Obviously “ $\leq_{\mathfrak{K}} \upharpoonright K_\lambda^*$  is a partial order”, so AxI, AxII hold and obviously AxV holds (see III.  600-0.2 ). The missing point was AxIII, about  $\leq_{\mathfrak{K}}$ -increasing union and it holds by 1.38(1). Then AxIV becomes easy by the definition of  $\leq_{\mathfrak{K}}^* = \leq_{\mathfrak{K}} \upharpoonright K_\lambda^*$  and lastly the amalgamation holds by 1.30.

2) By III§1 we can “lift  $\mathfrak{K}_\lambda^*$  up”, the result is  $\mathfrak{K}_{\geq \lambda}^\oplus$  (see III.  600-0.31 , III.  600-0.32 ).  $\square_{1.39}$

Let us formulate a major conclusion in ways less buried inside our notation.

**1.41 Conclusion.** Assume  $(\mathfrak{K}, \Phi)$  is pseudo solvable in  $\mu$ , then  $(\mathfrak{K}, \Phi)$  is pseudo solvable in  $\lambda$  provided that  $\text{LS}(\mathfrak{K}) < \lambda$ ,  $\mu = \mu^{< \lambda}$  (or just the hypothesis 1.18 holds),  $\text{cf}(\lambda) = \aleph_0$  and  $\lambda$  is an accumulation point of the class of the fix point of the sequence of the  $\beth$ 's.

*Proof.* By 1.39(1). □<sub>1.41</sub>

*Remark.* About [weak] solvability, see [Sh:F782].

## §2 TRYING TO ELIMINATE $\mu = \mu^{<\lambda}$

There was one point in §1 where we use  $\mu = \mu^\lambda$  (i.e. in 1.12, more accurately in justifying hypothesis 1.18(1)). In this section we try to eliminate it. So we try to prove  $M_1 \leq_{\mathfrak{K}_\mu} M_2 \Rightarrow M_1 \prec_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]} M_2$  for  $\theta < \lambda$ , hence we fix  $\mathfrak{K}, \mu, \theta$ . We succeed to do it with “few exceptions”.

**2.1 Hypothesis.** (We shall mention  $(b)_\mu$  or  $(b)_\mu^-$ ,  $(c)$ ,  $(d)$  when used! but not clause (a))

- (a)  $\mathfrak{K}$  is an a.e.c. and  $\Phi \in \Upsilon_{\mathfrak{K}}^{\text{or}}$
- (b) $_\mu$   $\mathfrak{K}$  categorical in  $\mu$  and  $\Phi \in \Upsilon_{\mathfrak{K}}^{\text{or}}$ , or at least
- (b) $_\mu^-$   $\mathfrak{K}$  is pseudo  $\mu$ -solvable as witnessed by  $\Phi \in \Upsilon_{\mathfrak{K}}^{\text{or}}$ , see Definition 1.4 in particular  $\text{EM}_{\tau(\mathfrak{K})}(I, \mu)$  is pseudo superlimit for  $I \in K_\lambda^{\text{lin}}$ ,
- (c)  $\mu \geq \beth_{1,1}(\text{LS}(\mathfrak{K}))$
- (d)  $\mu > \text{LS}(\mathfrak{K})$ .

**2.2 Convention:**  $K_\lambda^* = K_{\Phi, \lambda}^*$ , etc., see Definition 1.15.

**2.3 Definition.** Assume

$$\square \quad \mu \geq \chi \geq \theta > \text{LS}(\mathfrak{K})$$

1) We let  $K_{\mu, \chi}^1 = \{(M, N) : N \leq_{\mathfrak{K}} M, N \in K_\chi, M \in K_\mu \text{ and } \mu = \chi \Rightarrow M = N\}$  and let  $\leq_{\mathfrak{K}} = \leq_{\mathfrak{K}, \mu, \chi}$  be the following partial order on  $K_{\mu, \chi}$ ,  $(M_0, N_0) \leq_{\mathfrak{K}} (M_1, N_1)$  iff  $M_0 \leq_{\mathfrak{K}} M_1, N_0 \leq_{\mathfrak{K}} N_1$  (formally we should have written  $\leq_{\mathfrak{K}, \mu, \chi}$ ). Note that each pair  $(M, N) \in K_{\mu, \chi}$  determine  $\mu, \chi$ . So if  $\chi = \mu$ ,  $K_{\mu, \chi}$  is essentially  $\mathfrak{K}_\mu$ . Let  $K_\mu^1 = K_\mu$  and let  $\cup\{(M_i, N_i) : i < \delta\} = (\cup\{M_i : i < \delta\}, \cup\{N_i : i < \delta\})$  for any  $\leq_{\mathfrak{K}}$ -increasing sequence  $\langle (M_i, N_i) : i < \delta \rangle$ .

1A) Let  $K_{\mu, \chi} = K_{\mu, \chi}^2 = \{(M, N) \in K_{\mu, \chi}^1 : M \in K_\mu^*\}$  and  $K_\mu^2 = K_\mu^*$  but we use them only when  $\Phi$  witnesses  $\mathfrak{K}$  is pseudo  $\mu$ -solvable, i.e.  $(b)_\mu^-$  from Hypothesis 2.1 holds.

2) For  $k \in \{1, 2\}$  a formula  $\varphi(\bar{x}) \in \mathbb{L}_{\infty, \theta}[\mathfrak{K}]$  (so  $\ell g(\bar{x}) < \theta$ ), cardinal  $\kappa \geq \theta$  the main case being  $\kappa = \mu$ ; we may omit  $k$  if  $k = 2$ , and  $M \in K_\kappa^k, \bar{a} \in {}^{\ell g(\bar{x})}M$  we define when

$M \Vdash_k \varphi[\bar{a}]$  by induction on the depth of  $\varphi(\bar{x}) \in \mathbb{L}_{\infty, \theta}[\mathfrak{K}]$ , so the least obvious case is:

- (\*)  $M \Vdash_k (\exists \bar{y})\psi(\bar{y}, \bar{a})$  when for every  $M_1 \in K_\kappa^k$  such that  $M \leq_{\mathfrak{K}} M_1$  there is  $M_2 \in K_\kappa^k$  satisfying  $M_1 \leq_{\mathfrak{K}} M_2$  and  $\bar{b} \in {}^{\ell g(\bar{y})}M_2$  such that  $M_2 \Vdash_k \psi[\bar{b}, \bar{a}]$ .

Of course

- ( $\alpha$ ) for  $\varphi$  atomic,  $M \Vdash_k \varphi[\bar{a}]$  iff  $M \models \varphi[\bar{a}]$
- ( $\beta$ ) for  $\varphi(\bar{x}) = \bigwedge_{i < \alpha} \varphi_i(\bar{x})$  let  $M \Vdash_k \varphi[\bar{a}]$  iff  $M \Vdash_k \varphi_i[\bar{a}]$  for each  $i < \alpha$
- ( $\gamma$ )  $M \Vdash_k \neg \varphi[\bar{a}]$  iff for no  $N$  do we have  $M \leq_{\mathfrak{K}} N \in K_\kappa^k$  and  $N \Vdash_k \varphi[\bar{a}]$ .

3) Let  $k \in \{1, 2\}$ ,  $\Lambda \subseteq \mathbb{L}_{\infty, \theta}[\mathfrak{K}]$  (each formula with  $< \theta$  free variables, of course):

- (a)  $\Lambda$  is downward closed if it is closed under subformulas
- (b)  $\Lambda$  is  $(\mu, \chi)$ -model<sup>k</sup> complete (when  $\mu$  is clear from the context we may write  $\chi$ -model<sup>k</sup> complete) if  $|\Lambda| < \mu$ , and for every  $(M_0, N_0) \in K_{\mu, \chi}^k$  we can find  $(M, N) \in K_{\mu, \chi}^2$  above  $(M_0, N_0)$  which is  $\Lambda$ -generic, where:
- (c)  $(M, N) \in K_{\mu, \chi}^k$  is  $\Lambda$ -generic<sup>k</sup> when:  
if  $\varphi(\bar{x}) \in \Lambda$  and  $\bar{a} \in {}^{\ell g(\bar{x})}N$  then  
 $M \Vdash_k \varphi[\bar{a}] \Leftrightarrow N \models \varphi[\bar{a}]$  (yes! neither  $(M, N) \Vdash_k \varphi[\bar{a}]$  which was not defined, nor “ $M \models \varphi[\bar{a}]$ ”)
- (d)  $\Lambda$  is called  $(\mu, < \mu)$ -model<sup>k</sup> complete when  $|\Lambda| + \theta_\Lambda < \mu$  and for every  $\chi$ : if  $|\Lambda| + \theta_\Lambda \leq \chi < \mu$  then  $\Lambda$  is  $\chi$ -model<sup>k</sup> complete where  $\theta_\Lambda := \min\{\partial : \partial > \text{LS}(\mathfrak{K}) \text{ and } \Lambda \subseteq \mathbb{L}_{\infty, \partial}[\mathfrak{K}]\}$ . We say  $\Lambda$  is model<sup>k</sup> complete if it is  $(\mu, < \mu)$ -model<sup>k</sup> complete and  $\mu$  is understood from the context
- (e) above if  $\Phi$  or  $(\mathfrak{K}, \Phi)$  is not clear from the context we may replace  $\Lambda$  by  $(\Lambda, \Phi)$  or by  $(\Lambda, \Phi, \mathfrak{K})$ .

4) For  $M \in K_\kappa^k$ ,  $\bar{a} \in {}^{\theta}M$  and  $\Lambda \subseteq \mathbb{L}_{\infty, \theta}[\mathfrak{K}]$  let  $\text{gtp}_\Lambda^k(\bar{a}, \emptyset, M) = \{\varphi[\bar{a}] : M \Vdash_k \varphi[\bar{a}]\}$ ; if we write  $\theta$  instead of  $\Lambda$  we mean  $\mathbb{L}_{\infty, \theta}[\mathfrak{K}]$  (note: this type is not a priori complete) and we say that  $\bar{a}$  materializes this type in  $M$ . To stress  $\kappa$  we may write  $\text{gtp}_\Lambda^{\kappa, k}(\bar{a}, \emptyset, M)$  or  $\text{gtp}_\theta^{\kappa, k}(\bar{a}, \emptyset, M)$  though  $M$  determines  $\kappa$ .

5) We say  $M \in K_\kappa$  is  $\Lambda$ -generic<sup>k</sup> when for every  $\varphi(\bar{x}) \in \Lambda$  and  $\bar{a} \in {}^{\ell g(\bar{x})}M$  we have  $M \Vdash_k \varphi[\bar{a}] \Leftrightarrow M \models \varphi[\bar{a}]$ . So  $M \in K_\mu^k$  is  $\Lambda$ -generic<sup>k</sup> iff  $(M, M) \in K_{\mu, \mu}^k$  is  $\Lambda$ -generic<sup>k</sup>. We say  $\Lambda$  is  $\kappa$ -model<sup>k</sup> complete when every  $M \in K_\kappa^k$  has a  $\Lambda$ -generic  $\leq_{\mathfrak{K}}$ -extension in  $K_\kappa^k$  (so depend on  $\mathfrak{K}$  and if  $k = 2$  also on  $\Phi$ ).

6) In all cases above, if  $k = 2$  we may omit it.

**2.4 Claim.** Assume that  $\text{LS}(\mathfrak{K}) < \theta \leq \chi < \mu$  and  $\kappa > \theta$  and  $k \in \{1, 2\}$  so if  $k = 2$  then  $2.1(b)_\mu^-$  holds, see 2.3(1A).

1)  $(K_{\mu, \chi}^k, \leq_{\mathfrak{K}})$  is a partial order and chains of length  $\delta < \chi^+$  of members has a  $\leq_{\mathfrak{K}}$ -lub, this is the union, see 2.3(1). If  $\text{EM}_{\tau(\mathfrak{K})}(\mu, \Phi)$  is superlimit (not just pseudo superlimit) then  $K_{\mu, \chi}^2$  is a dense subclass of  $K_{\mu, \chi}^1$  under  $\leq_{\mathfrak{K}}$ .

2) If  $M_1 \Vdash_k \varphi(\bar{a})$  and  $M_1 \leq_{\mathfrak{K}} M_2$  are from  $K_{\kappa}^k$  then  $M_2 \Vdash_k \varphi[\bar{a}]$ .

3) If  $(M_\ell, N_\ell) \in K_{\mu, \chi}^k$  are  $\Lambda$ -generic<sup>k</sup> for  $\ell = 1, 2$  and  $(M_1, N_1) \leq_{\mathfrak{K}} (M_2, N_2)$  then  $N_1 \prec_{\Lambda} N_2$ .

4) If  $M_i \in K_{\kappa}^k$  for  $i < \delta$  is  $\leq_{\mathfrak{K}}$ -increasing,  $\delta < \kappa^+$ ,  $\text{cf}(\delta) \geq \theta$ ,  $\Lambda \subseteq \mathbb{L}_{\infty, \theta}[\mathfrak{K}]$  and each  $M_i$  is  $\Lambda$ -generic<sup>k</sup>, then  $M_\delta := \bigcup_{i < \delta} M_i$  is  $\Lambda$ -generic<sup>k</sup> and  $i < \delta \Rightarrow M_i \prec_{\Lambda} M_\delta$ .

5) If  $(M_i, N_i) \in K_{\mu, \chi}^k$  for  $i < \delta$  is  $\leq_{\mathfrak{K}}$ -increasing,  $\delta < \chi^+$ ,  $\text{cf}(\delta) \geq \theta$ ,  $\Lambda \subseteq \mathbb{L}_{\infty, \theta}[\mathfrak{K}]$  and each  $(M_i, N_i)$  is  $\Lambda$ -generic<sup>k</sup>, then  $(\bigcup_{i < \delta} M_i, \bigcup_{i < \delta} N_i)$  is  $\Lambda$ -generic<sup>k</sup> and  $N_j \prec_{\Lambda}$

$\bigcup_{i < \delta} N_i$  for each  $j < \delta$ .

*Proof.* Should be clear; in part (1) for  $k = 2$  we use clause  $(b)_\mu^-$  of 2.1. In part (5) note that  $\cup\{M_i : i < \delta\} \in K_\mu^*$  by Clause  $(b)_\mu^-$  of 2.1.  $\square_{2.4}$

**2.5 Exercise:** If  $(M, N)$  is  $\Lambda$ -generic<sup>k</sup> and  $(M, N) \leq_{\mathfrak{K}} (M', N) \in K_{\mu, \chi}^k$  then  $(M', N)$  is  $\Lambda$ -generic<sup>k</sup>.

**2.6 Claim.** Assume that  $\mu \geq \chi \geq \theta > \text{LS}(\mathfrak{K})$  and  $k \in \{1, 2\}$ .

1) The set of quantifier free formulas in  $\mathbb{L}_{\infty, \theta}[\mathfrak{K}]$  is  $(\mu, \chi)$ -model<sup>k</sup> complete.

2) If  $\Lambda_\varepsilon \subseteq \mathbb{L}_{\infty, \theta}(\tau_{\mathfrak{K}})$  is downward closed,  $(\mu, \chi)$ -model<sup>k</sup> complete for  $\varepsilon < \varepsilon^*$ , and  $\Lambda := \bigcup_{\varepsilon < \varepsilon^*} \Lambda_\varepsilon$ ,  $\theta = \text{cf}(\theta) \leq \chi \vee \theta < \chi$ ,  $\varepsilon^* < \chi^+$  (and  $\mu > \theta \vee \mu = \theta = \text{cf}(\theta)$ ) then  $\Lambda$  is  $(\mu, \chi)$ -model<sup>k</sup> complete.

*Proof.* 1) Easy.

2) Given  $(M, N) \in K_{\mu, \chi}^k$  let  $\theta_r$  be  $\min\{\partial : \partial \geq \theta \text{ is regular}\}$ . Clearly  $\theta_r \leq \chi$  and we choose  $(M_i, N_i) \in K_{\mu, \chi}^k$  for  $i \leq \varepsilon^* \times \theta_r$  such that

- ⊗ (a)  $\langle M_i : i \leq \varepsilon^* \times \theta_r \rangle$  is  $\leq_{\mathfrak{K}}$ -increasing continuous
- (b)  $\langle N_i : i \leq \varepsilon^* \times \theta_r \rangle$  is  $\leq_{\mathfrak{K}}$ -increasing continuous
- (c) if  $i = \varepsilon^* \times \gamma + \varepsilon$  and  $\varepsilon < \varepsilon^*$  then  $(M_{i+1}, N_{i+1})$  is  $\Lambda_\varepsilon$ -generic<sup>k</sup>
- (d)  $(M_0, N_0) = (M, N)$ .

There is no problem to do this.

Now for each  $\varepsilon < \varepsilon^*$  the sequence  $\langle (M_{\varepsilon^* \times \gamma + \varepsilon + 1}, N_{\varepsilon^* \times \gamma + \varepsilon + 1}) : \gamma < \theta_r \rangle$  is  $\leq_{\mathfrak{K}, \mu, \chi^-}$  increasing with union  $(M_{\varepsilon^* \times \theta_r}, N_{\varepsilon^* \times \theta_r})$ , and each member of the sequence is  $\Lambda_\varepsilon$ -generic<sup>k</sup> hence by 2.4(5) we know that the pair  $(M_{\varepsilon^* \times \theta_r}, N_{\varepsilon^* \times \theta_r})$  is  $\Lambda_\varepsilon$ -generic<sup>k</sup>. As this holds for each  $\Lambda_\varepsilon$  it holds for  $\Lambda$  so  $(M_{\varepsilon^* \times \theta_r}, N_{\varepsilon^* \times \theta_r})$  is as required.  $\square_{2.6}$

From now on in this section

*2.7 Hypothesis.* We assume (a) + (b) <sub>$\mu^-$</sub>  of 2.1 and we omit  $k$  using Definition 2.3 meaning  $k = 2$ .

**2.8 Claim.** 1) For  $M \in K_\mu^*$  and  $\text{LS}(\mathfrak{K}) < \theta < \mu$  the number of complete  $\mathbb{L}_{\infty, \theta}[\mathfrak{K}]$ -types realized by sequences from  ${}^\theta M$  is  $\leq 2^{<\theta}$ , moreover, the relation  $\mathcal{E}_M^{<\theta} := \{(\bar{a}, \bar{b}) : \bar{a}, \bar{b} \in {}^\theta M \text{ and some automorphism of } M \text{ maps } \bar{a} \text{ to } \bar{b}\}$  is an equivalence relation with  $\leq 2^{<\theta}$  equivalence classes.

2) Hence there is a set  $\Lambda_* = \Lambda_\theta^* = \Lambda_{\mathfrak{K}, \Phi, \mu, \theta}^* \subseteq \mathbb{L}_{\infty, \theta}[\mathfrak{K}]$  such that:

- (a)  $|\Lambda_*| \leq 2^{<\theta}$  and  $\Lambda_* \subseteq \mathbb{L}_{(2^{<\theta})^+, \theta}[\mathfrak{K}]$
- (b)  $\Lambda_*$  is closed under sub-formulas and finitary operations
- (c) each  $\varphi(\bar{x}) \in \Lambda_*$  has quantifier depth  $< \gamma^*$  for some  $\gamma^* < (2^{<\theta})^+$
- (d) for  $\alpha < \theta$ ,  $M \in K_\mu^*$  and  $\bar{a} \in {}^\alpha M$ , the  $\Lambda_*$ -type which  $\bar{a}$  realizes in  $M$  determines the  $\mathbb{L}_{\infty, \theta}[\mathfrak{K}]$ -type which  $\bar{a}$  realizes in  $M$ , moreover one formula in the type determine it
- (e) similarly for materialize in  $M \in K_\mu^*$ , see Definition 2.3(4)
- (f) if  $\text{LS}(\mathfrak{K}) \leq \chi < \mu$  and  $(M, N) \in K_{\mu, \chi}$  is  $\Lambda_*$ -generic then it is  $\mathbb{L}_{\infty, \theta}[\mathfrak{K}]$ -generic
- (g) if  $M \in K_\mu^2$  is  $\Lambda_*$ -generic then it is  $\mathbb{L}_{\infty, \theta}[\mathfrak{K}]$ -generic.

*Remark.* Part (1) can also be proved using just  $(\lambda + 1) \times I_*$  with  $I_*$  a  $\theta$ -saturated dense linear order with neither first nor last element, but this is not clear for 2.11(1).

*Proof.* 1) By 5.1(1) and categoricity of  $K_\lambda^*$ .

2) Follows but we elaborate.

Let  $\{\bar{a}_\alpha : \alpha < \alpha^* \leq 2^{<\theta}\}$  be a set of representatives of the  $\mathcal{E}_M^{<\theta}$ -equivalence classes. For each  $\alpha \neq \beta$  such that  $\ell g(\bar{a}_\alpha) = \ell g(\bar{a}_\beta)$ , let  $\bar{x}_\alpha = \langle x_i : i < \ell g(\bar{a}_\alpha) \rangle$  and choose  $\varphi_{\alpha, \beta}(\bar{x}_\alpha), \psi_{\alpha, \beta}(\bar{x}_\alpha) \in \mathbb{L}_{(2^{<\theta})^+, \theta}[\mathfrak{K}]$  such that, if possible we have  $M \models \varphi_{\alpha, \beta}[\bar{a}_\alpha] \wedge \neg \varphi_{\alpha, \beta}[\bar{a}_\beta]$  and, under this, if possible  $M \models \psi_{\alpha, \beta}(\bar{a}_\alpha) \wedge \neg \psi_{\alpha, \beta}(\bar{a}_\beta)$  but in any case  $M \models \varphi_{\alpha, \beta}[\bar{a}_\alpha]$  and  $M \models \psi_{\alpha, \beta}[\bar{a}_\alpha]$ . Let  $\varphi_\alpha(\bar{x}) = \bigwedge \{\varphi_{\alpha, \beta}(\bar{x}_\alpha) : \beta <$



$\alpha^*, \beta \neq \alpha$  and  $\ell g(\bar{a}_\beta) = \ell g(\bar{a}_\alpha)$  and similarly  $\psi_\alpha(\bar{x}_\alpha)$ . Let  $\Lambda_*$  be the closure of  $\{\varphi_{\alpha,\beta}, \psi_{\alpha,\beta}, \varphi_\alpha, \psi_\alpha : \alpha \neq \beta < \alpha^*\}$  under subformulas and finitary operations. Obviously, clauses (a),(b) hold hence the existence of  $\gamma^* < (2^{<\theta})^+$  as required in clause (c) follows. Clause (d) holds as  $\bar{a} \mathcal{E}_M^{\leq \theta} \bar{b} \Rightarrow \text{tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar{a}, \emptyset, M) = \text{tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar{b}, \emptyset, M)$  using the automorphisms and for  $\alpha, \beta < \alpha_*$  such that  $\ell g(\bar{a}_\alpha) = \ell g(\bar{a}_\beta)$  we have  $M \models (\forall \bar{x}_\alpha)(\varphi_\alpha(\bar{x}_\alpha) = \varphi_\beta(\bar{x}_\beta))$  implies  $\text{tp}_{\mathbb{L}_{(2^{<\theta})^+,\theta}[\mathfrak{K}]}(\bar{a}_\alpha, \emptyset, M) = \text{tp}_{\mathbb{L}_{(2^{<\theta})^+,\theta}[\mathfrak{K}]}(\bar{a}_\beta, \emptyset, M)$  and even  $\text{tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar{a}_\alpha, \emptyset, M) = \text{tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar{a}_\beta, \emptyset, M)$  recalling the choice of the  $\varphi_{\alpha,\beta}$ 's.

Clause (e) holds similarly by the choice of the  $\psi_{\alpha,\beta}$ 's. Clauses (f),(g) should also be clear. (The proof is similar to the proof of the classical 0.17(3).)  $\square_{2.8}$

**2.9 Observation.** Assume (2.1(b) $^-_\mu$  of course and)  $\Lambda \subseteq \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$  and  $\mu > 2^{<\theta}$  and  $\theta > \text{LS}(\mathfrak{K})$ .

1) The number of complete  $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$ -types realized in some  $M \in K_\mu^*$ , by a sequence of length  $< \theta$  of course, is  $\leq 2^{<\theta}$ . Hence every formula in  $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$  is equivalent, for models from  $K_\mu^*$  to a formula of quantifier depth  $< (2^{<\theta})^+$ , even from  $\Lambda_* \subseteq \mathbb{L}_{(2^{<\theta})^+,\theta}[\mathfrak{K}]$  where  $\Lambda_*$  is in 2.8(2).

2) Assume that  $I_1 \subseteq I_2$  are well ordered,  $\text{cf}(I_1), \text{cf}(I_2) > 2^{<\theta}$  and  $t \in I_2 \setminus I_1 \Rightarrow 2^{<\theta} < \text{cf}(I_1 \upharpoonright \{s \in I_1 : s <_{I_2} t\})$  and  $t \in I_2 \setminus I_1 \Rightarrow 2^{<\theta} < \text{cf}(I_2 \upharpoonright \{s \in I_2 : (\forall r \in I_1)(r <_{I_2} t \equiv r <_{I_2} s)\})$ . Then  $\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \prec_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} \text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$ .

3) If  $M = \text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ ,  $|I| = \mu$ ,  $I$  well ordered of cofinality  $> 2^{<\theta}$ ,  $\bar{a} \in {}^\alpha M$  where  $\alpha < \theta$  and  $a_i = \sigma_i(\dots, a_{t_{i,\ell}}, \dots)_{\ell < n(i)}$  for  $i < \alpha$  then  $\text{tp}_{\Lambda_*}(\bar{a}, \emptyset, M)$  is determined by  $\langle \sigma_i(x_0, \dots, x_{n(\ell)-1}) : i < \ell g(\bar{a}) \rangle$  and the essential  $\theta$ -type of  $\langle t_{i,\ell} : i < \ell g(\bar{a}), \ell < n(i) \rangle$ , see Definition 2.10 below.

Before proving 2.9

**2.10 Definition.** 1) For  $\bar{t} = \langle t_i : i < \alpha \rangle \in {}^\alpha I$ ,  $I$  well ordered, let the essential  $\theta$ -type of  $\bar{t}$  in  $I$  be the essential  $(\theta, (2^{<\theta})^+)$ -type where for an ordinal  $\gamma$  we let the essential  $(\theta, \gamma)$ -type of  $\bar{t}$  in  $I$ ,  $\text{estp}_{\theta,\gamma}(\bar{t}, \emptyset, I)$  be the following information stipulating  $t_\alpha = \infty$ :

- (a) the truth value of  $t_i < t_j$  (for  $i, j < \alpha$ )
- (b)  $\text{otp}([r_i, t_i)_I)$  for  $i < \alpha$  where for  $i \leq \alpha$  we let  $r_i$  be the minimal member  $r$  of  $I$  such that  $\text{otp}([r, t_i)_I) < \theta \times \gamma$  and  $r \leq_I t_i$  and  $j < \alpha \wedge t_j < t_i \Rightarrow t_j \leq r$
- (c)  $\text{Min}\{\theta \times \gamma, \text{otp}[s_i, r_i)_I\}$  for  $i \leq \alpha$  where we let  $s_i$  be the minimal member of  $I$  such that  $(\forall j < \alpha)[t_j <_I t_i \Rightarrow t_j <_I s_i]$
- (d)  $\text{Min}\{\theta, \text{cf}(I \upharpoonright \{s : s <_I r_i\})\}$  for  $i \leq \alpha$  which may be zero.

2) Let the function implicit in 2.9(3) be called  $\mathbf{t}_\Lambda^\mu = \mathbf{t}_{\mathfrak{K},\Lambda}^\mu = \mathbf{t}_{\mathfrak{K},\Phi,\Lambda}^\mu$ , i.e.,  $\mathbf{t}_\Lambda^\mu(\mathbf{s}, \bar{\sigma}) = \text{tp}_\Lambda(\bar{a}, \emptyset, M)$  when  $\bar{a} = \langle \sigma_i(\dots, a_{t_{\beta(i,\ell)}}, \dots)_{\ell < n_i} : i < \ell g(\bar{a}) \rangle$ ,  $\bar{\sigma} = \langle \sigma_i(\dots, x_{\beta(i,\ell)}), \dots \rangle_{\ell < n; i < \ell g(\bar{a})}$

$\ell g(\bar{a})$  and  $\mathbf{s}$  is the essential  $\theta$ -type of  $\langle t_{i,\ell} : i < \ell g(\bar{a}), \ell < n_i \rangle$  in  $I$ .  
If  $\Lambda = \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$  we may write just  $\theta$ .

*Proof of 2.9.* 1) By 2.8(1) this holds for each  $M \in K_\mu^*$ .

2) It is known by Kino [Kin66] that  $I_1 \prec_{\mathcal{L}} I_2$  if  $\mathcal{L} \subseteq \{\varphi \in \mathbb{L}_{\infty,\theta}(\{<\}) : \varphi \text{ has quantifier depth } < (2^{<\theta})^+\}$ . From this the result follows by part (1).

More fully let  $\theta_r$  be the first regular cardinal  $\geq \theta$ , and we say that the pair  $(I_1, I_2)$  is  $\gamma$ -suitable when we replace in the assumptions “of cofinality  $> 2^{<\theta}$ ” by “of cofinality  $\geq \theta$  and of order type divisible by  $\theta \times \gamma$ ”. Now we prove by induction on  $\gamma$  that

- ⊙<sub>1</sub> assume that for  $\alpha < \theta$  and for  $\ell = 1, 2$  we have:  $I_\ell$  is a well ordering,  $\bar{t}^\ell = \langle t_i^\ell : i < \alpha \rangle$  is  $<_{I_\ell}$ -increasing,  $t_0^\ell$  is the first element of  $I_\ell$ , we stipulate  $t_\alpha^\ell = \infty$  and  $\text{otp}([t_i^\ell, t_{i+1}^\ell)_{I_0}) = \theta_r \gamma \alpha_i^\ell + \beta_i$  where  $\beta_i < \theta \gamma$  and  $(\text{cf}(\alpha_i^1) = \text{cf}(\alpha_i^1)) \vee (\text{cf}(\alpha_i^1) \geq \theta \wedge \text{cf}(\alpha_i^2) \geq \theta)$ .  
Then for any formula  $\varphi(\langle x_i : i < \alpha \rangle) \in \mathbb{L}_{\infty,\theta}(\{<\})$  of quantifier depth  $\leq \gamma$  we have  $I_1 \models \varphi[\bar{t}^1] \Leftrightarrow I_2 \models \varphi[\bar{t}^2]$ .

Hence

- ⊙<sub>2</sub> if  $\vartheta(\bar{x}) \in \mathbb{L}_{\infty,\theta}(\{<\})$  has quantifier depth  $< \gamma$  and  $(I_1, I_2)$  is  $\gamma$ -suitable and  $\bar{t} \in {}^{\ell g(\bar{x})} I_1$  then  $I_1 \models \varphi[\bar{t}] \Leftrightarrow I_2 \models \varphi[\bar{t}]$ .

3) Follows by part (2). □<sub>2.9</sub>

## 2.11 Claim. Assume

- (a)  $M \in K_\mu^*$   
 (b)  $\Lambda \subseteq \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$  is downward closed,  $|\Lambda| \leq \chi$ ,  $\text{LS}(\mathfrak{K}) < \theta \leq \chi < \mu$  and  $2^{<\theta} \leq \chi$  and  $\theta = \text{cf}(\theta) \vee \theta < \chi$  so  $\Lambda = \Lambda_*$  from 2.8 is O.K.  
 (c) in part (3), (4), (5) we assume  $(\chi^{<\theta} \leq \mu) \vee (\text{cf}(\mu) \geq \theta)$   
 (d) for part (6) we assume  $\text{cf}(\mu) \geq \theta$  (hence the demand in clause (c) holds).

1) If  $M \in K_\mu^*$  then  $\{\text{gtp}_\Lambda(\bar{a}, \emptyset, M) : \bar{a} \in {}^{\theta>} M\}$  has cardinality  $\leq 2^{<\theta}$ .

2) If  $(M, N) \in K_{\mu,\chi}$  then we can find  $N'$ ,  $(M, N) \leq_{\mathfrak{K}} (M, N') \in K_{\mu,\chi}$  such that

- (\*) if  $\alpha < \theta$  and  $\bar{b} \in {}^\alpha M$  and  $\Lambda \subseteq \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$  then for some  $\bar{b}' \in {}^\alpha(N')$  we have:  
for every  $\bar{a} \in {}^{\theta>} N$ ,  $\text{gtp}_\Lambda(\bar{a} \hat{\ } \bar{b}, \emptyset, M) = \text{gtp}_\Lambda(\bar{a} \hat{\ } \bar{b}', \emptyset, M)$ .

3) If  $(M, N) \in K_{\mu, \chi}$ , then we can find  $(M_1, N_1)$  such that  $(M, N) \leq_{\mathfrak{K}} (M_1, N_1) \in K_{\mu, \chi}$  and (note that  $\bar{y}$  may be the empty sequence)

(\*) if  $\exists \bar{y} \varphi(\bar{y}, \bar{x}) \in \Lambda$  and  $\bar{a} \in {}^{\ell g(\bar{x})}N$  then  $M_1 \models \neg \exists \bar{y} \varphi(\bar{y}, \bar{x})$  or for some  $\bar{b} \in {}^{\ell g(\bar{y})}(N_1)$  we have  $M_1 \models \varphi[\bar{b}, \bar{a}]$ .

4) In part (3) we can demand

(\*)<sup>+</sup> if  $\exists \bar{y} \varphi(\bar{y}, \bar{x}) \in \Lambda$  and  $\bar{a} \in {}^{\ell g(\bar{x})}(N_1)$  then  $M_1 \models \neg(\exists \bar{y}) \varphi(\bar{y}, \bar{x})$  or for some  $\bar{b} \in {}^{\ell g(\bar{y})}(N_1)$  we have  $M_1 \models \varphi[\bar{b}, \bar{a}]$ .

5) In part (4) it follows that the pair  $(M_1, N_1)$  is  $\Lambda$ -generic (most interesting for  $\Lambda_*$ , see 2.8).

6) If  $M_1 \in K_{\mu}^*$  then it is  $\Lambda$ -generic.

*Proof.* 1) Proved just like 2.8(1).

2) First assume  $\theta$  is a successor cardinal. As  $M \in K_{\mu}^*$  without loss of generality  $M = \text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$  for some linear order  $I$  of cardinality  $\mu$  as in 5.1(1),(4) with  $\theta^-, \theta, \chi^+, \mu$  here standing for  $\mu, \theta_1, \theta_2, \lambda$  there. It follows that for some  $J \subseteq I$  of cardinality  $\chi$  we have  $N \subseteq \text{EM}_{\tau(\mathfrak{K})}(J, \Phi)$ , and let  $J^+ \subseteq I$  be such that  $J \subseteq J^+$ ,  $|J^+ \setminus J| = \chi$  and for every  $\bar{t} \in {}^{\theta}I$  there is an automorphism  $f$  of  $I$  over  $J$  which maps  $\bar{t}$  to some member of  ${}^{\ell g(\bar{t})}(J^+)$ .

Lastly, let  $N' = \text{EM}_{\tau(\mathfrak{K})}(J^+, \Phi)$ , it is easy to check (see 1.4) that (\*) holds. If  $\theta$  is a limit ordinal it is enough to prove for each  $\partial < \theta$ , a version of (\*) with  $\alpha < \partial$ ; and this gives  $N'_{\partial}$ . Now we choose  $N'$  such that  $\partial < \theta \Rightarrow N'_{\partial} \leq_{\mathfrak{K}} N'$  and  $(M, N') \in K_{\mu, \chi}$ .

3),4),5),6) We prove by induction on  $\gamma$  that if we let  $\Lambda_{\gamma}$  be  $\{\varphi(\bar{x}) : \varphi(\bar{x}) \in \Lambda \text{ has quantifier depth } < 1 + \gamma\}$  then parts (3),(4),(5),(6) holds for  $\Lambda_{\gamma}$ . For all four parts,  $|\Lambda| \leq \chi$  hence  $|\Lambda_{\gamma}| \leq \chi$  and it suffices to consider  $\gamma < \chi^+$ . For  $\gamma = 0$  they are trivial and for  $\gamma$  limit also easy (let  $\theta_r$  be the first regular  $\geq \theta$  and extend  $|\gamma|^+ \times \theta_r$  times taking care of  $\Lambda_{\beta}$  in stage  $\gamma \times \zeta + \beta$  for each  $\beta < \gamma$ ). So let  $\gamma = \beta + 1$ .

We first prove (3), but we have two cases (see clause (c)) of the assumption. If  $\chi^{<\theta} \leq \mu$  this is straight by bookkeeping. So assume  $\text{cf}(\mu) \geq \theta$ . Given  $(M, N) \in K_{\mu, \chi}$  we try to choose by induction on  $i < \chi^+$  a pair  $(M_i, N_i)$  and for  $i$  odd also  $\psi_i(\bar{y}_i, \bar{x}_i), \bar{a}_i, \bar{b}_i$  such that

- ⊗<sub>1</sub> (a)  $(M_0, N_0) = (M, N)$
- (b)  $(M_i, N_i) \in K_{\mu, \chi}$  is  $\leq_{\mathfrak{K}}$ -increasing continuous
- (c)  $M_{i+1}$  is  $\Lambda_{\beta}$ -generic for  $i$  even
- (d) for  $i$  odd  $\psi_i(\bar{y}_i, \bar{x}_i) \in \Lambda_{\beta}$  and  $\bar{a}_i \in {}^{\theta}N$  and  $\bar{b}_i \in {}^{\theta}(N_{i+1})$  are such

- that  $\ell g(\bar{a}_i) = \ell g(\bar{x}_i), \ell g(\bar{b}_i) = \ell g(\bar{y}_i)$  and
- ( $\alpha$ )  $\bar{b} \in {}^{\ell g(\bar{y}_i)}(M_i) \Rightarrow M_i \not\models \psi_i[\bar{b}_i, \bar{a}]$  but
  - ( $\beta$ )  $M_{i+1} \models \psi_i[\bar{b}_i, \bar{a}_i]$
  - ( $\gamma$ ) for every  $\bar{b} \in {}^{\theta >}(M_{i+1})$  there is an automorphism of  $M_{i+1}$  over  $N_i$  mapping  $\bar{b}$  into  $N_{i+1}$ .

If we succeed, by part (2) applied to the pair of models  $(\bigcup_{i < \chi^+} M_i, N)$  as  $\chi^+ \leq \mu$

this pair belongs to  $K_{\mu, \chi}$  we get  $N'$  as there, hence for some odd  $i < \chi^+$ ,  $N' \subseteq M_i$ , let  $\zeta = i + 2$  and this gives a contradiction to the choice of  $(\psi_\zeta, \bar{a}_\zeta, \bar{b}_\zeta)$ .

[Why? There is an automorphism  $f$  of  $M := \bigcup \{M_j : j < \chi^+\}$  over  $N$  mapping  $\bar{b}_\zeta$  into  $N'$  hence into  $M_i$  hence  $f(\bar{b}_\zeta) \in {}^{\theta >}(M_\zeta)$ . We know (by clause (d)( $\beta$ ) above) that  $M_{\zeta+1} \models \psi_\zeta[\bar{b}_\zeta, \bar{a}_\zeta]$  but  $M_{\zeta+1} \leq_{\bar{\kappa}_\mu} M$  hence  $M \models \psi_\zeta[\bar{a}_\zeta, \bar{b}_\zeta]$ . Recall that  $f$  is an automorphism of  $M$  over  $N$  hence  $M \models \psi_\zeta[f(\bar{b}_\zeta), f(\bar{a}_\zeta)]$ , but  $\bar{a}_\zeta \in {}^{\theta >}N$  so  $f(\bar{a}_\zeta) = \bar{a}_\zeta$  hence  $M \models \psi_\zeta[\bar{b}_\zeta, f(\bar{a}_\zeta)]$  but  $M_\zeta \leq_{\bar{\kappa}_\mu} M$  and  $\bar{a}, f(\bar{b}_\zeta)$  are from  $M_\zeta$  hence  $M_\zeta \not\models \neg \psi_\zeta[f(\bar{b}_\zeta), (\bar{a}_\zeta)]$ . However by clause (d)( $\alpha$ ) of  $\otimes_1$  we have  $M_\zeta \not\models \psi_\zeta[f(\bar{b}_\zeta), \bar{a}_\zeta]$ . But as  $i$  is an odd ordinal the last two sentences contradicts clause (c) of  $\otimes_1$  applied to  $i + 1$ .]

Hence we are stuck for some  $i < \chi^+$ . Now for  $i = 0$  clause  $\otimes(a)$  gives a permissible value and for  $i$  limit take unions noting that clauses (c),(d) required nothing. So  $i = j + 1$ ; if  $j$  is even we apply the induction hypothesis to part (6) for the pair  $(M_i, N_i)$ . Hence  $j$  is odd so we cannot choose  $\psi_j(\bar{y}, \bar{x}), \bar{a}_j, \bar{b}_j$ , recalling part (2) so the pair  $(M_j, N_j)$  is as required thus proving (3) (for  $\Lambda_\gamma$ ).

Second, we prove part (4). We can now again try to choose by induction on  $i < \chi^+$  a pair  $(M_i, N_i)$  satisfying

- $\otimes_2$  (a)  $(M_0, N_0) = (M, N)$
- (b)  $(M_i, N_i) \in K_{\mu, \chi}$  is  $\leq_{\bar{\kappa}}$ -increasing continuous
- (c) if  $i = 2j + 1$ , then  $(M_{i+1}, N_{i+1})$  is as in part (3) for  $\Lambda_\gamma$  with  $(M_i, N_i), (M_{i+1}, N_{i+1})$  here standing for  $(M, N), (M_1, N_1)$  there
- (d) if  $i = 2j$  then for some  $\psi_i(\bar{y}_i, \bar{x}_i) \in \Lambda_\beta$  and  $\bar{a}_i \in ({}^{\ell g(\bar{x}_i)})(N_i)$  and  $\bar{b}_i \in ({}^{\ell g(\bar{y}_i)})(N_{i+1})$  we have  $M_{i+1} \models \psi_i(\bar{b}_i, \bar{a}_i)$  but  $\bar{b} \in {}^{\ell g(\bar{y}_i)}(M_i) \Rightarrow M_i \not\models \psi_i[\bar{b}, \bar{a}_i]$ .

If we succeed, let  $S_0 = \{\delta < \chi^+ : \text{cf}(\delta) \geq \theta\}$ , so by an assumption  $S$  is a stationary subset of  $\chi^+$ , i.e. as by clause  $\Box(b)$  we have  $\theta = \text{cf}(\theta) \leq \chi \vee \theta < \chi$ ; also for  $\delta \in S_0$ , as  $\langle N_i : i < \delta \rangle$  is increasing with union  $N_\delta$ , and  $\delta = 2\delta$  clearly  $\bar{a}_\delta$  is well defined, so for some  $i(\delta) < \delta$  we have  $\bar{a}_\delta \in {}^{\theta >}(N_{i(\delta)})$  and without loss of generality  $i(\delta) = 2j(\delta) + 1$  for some  $j(\delta)$  hence by clause (c) of  $\otimes_2$  the pair  $(M_{i(\delta)+1}, N_{i(\delta)+1})$  is as required

there contradiction as in the proof for part (3). Hence for some  $i$  we cannot choose  $(M_i, N_i)$ .

For  $i = 0$  let  $(M_i, N_i) = (M, N)$  so only clauses (a) + (b) of  $\otimes_2$  apply and are satisfied. For  $i$  limit take unions. So  $i = j + 1$ . If  $j = 1 \bmod 2$ , clause (d) of  $\otimes_2$  is relevant and we use part (3) for  $\Lambda_\beta$  which holds as we have just proved it.

Lastly, if  $j = 2 \bmod 2$  and we are stuck then the pair  $(M_j, N_j)$  is as required. Third, Part (5) should be clear but we elaborate.

We prove by induction on  $\gamma'$  that if  $\varphi(\bar{x}) \in \Lambda_\gamma$  has quantifier depth  $< 1 + \gamma'$  then for every  $\bar{a} \in {}^{\ell g(\bar{x})}(N_1)$  we have  $M_1 \models \varphi[\bar{a}] \Leftrightarrow N_1 \models \varphi[\bar{a}]$ . For atomic  $\varphi$  this is obvious and for  $\varphi = \bigwedge_{i < \alpha} \varphi_i$  should be clear. If  $\varphi(\bar{x}) = \neg\psi(\bar{x})$  note that in  $(*)^+$  of part (4) we can use empty  $\bar{y}$  so  $\neg(\exists \bar{y})\psi(\bar{x}) = \neg\psi(\bar{x})$ . Also for  $\varphi(\bar{x}) = (\exists \bar{y})\varphi'(\bar{y}, \bar{x})$  we apply part (4).

Fourth, we deal with part (6), so (see clause (d) of the assumption) we have  $\text{cf}(\mu) \geq \theta$ . Let  $\chi = \langle \chi_i : i < \text{cf}(\mu) \rangle$  be constantly  $\mu^-$  (so  $\mu = \chi_i^+$ ) if  $\mu$  is a successor cardinal, and be increasing continuous with limit  $\mu$ ,  $2^{<\theta} < \chi_i < \mu$  if  $\mu$  is a limit cardinal recalling  $2^{<\theta} < \mu$  by  $\square(b)$ . Consider  $K_{\mu, \bar{\chi}} = \{\bar{M} : \bar{M} = \langle M_i : i \leq \text{cf}(\mu) \rangle \text{ is } \leq_{\bar{\mathfrak{K}}}\text{-increasing continuous, } M_{\text{cf}(\mu)} \in K_\mu^* \text{ and } M_i \in K_{\chi_i} \text{ for } i < \text{cf}(\mu)\}$  ordered by  $\bar{M}^1 \leq_{\bar{\mathfrak{K}}} \bar{M}^2 \iff i \leq \text{cf}(\mu) \Rightarrow M_i^1 \leq_{\bar{\mathfrak{K}}} M_i^2$ .

By 2.11 and part (5) for  $\Lambda_\gamma$  which we proved we can easily find  $\bar{M} \in K_{\mu, \bar{\chi}}$  such that  $i < \text{cf}(\mu) \Rightarrow (M_{\text{cf}(\mu)}, M_{i+1})$  is  $\Lambda_\gamma$ -generic; such  $\bar{M}$  we call  $\Lambda_*$ -generic. Next

☐ if  $\varphi(\bar{x}) \in \Lambda_\gamma$  and  $\bar{M}$  is  $\Lambda_\gamma$ -generic,  $\bar{a} \in {}^{\theta >}(M_i)$ ,  $i$  successor,  $\varphi(\bar{x}) \in \mathbb{L}_{\infty, \theta}[\bar{\mathfrak{K}}]$  and  $\ell g(\bar{x}) = \ell g(\bar{a})$  then  $M_{\text{cf}(\mu)} \models \varphi[\bar{a}] \Leftrightarrow M_{\text{cf}(\mu)} \Vdash \varphi[\bar{a}]$ .

[Why? Recalling  $\text{cf}(\mu) \geq \theta$ , we prove this by induction on the quantifier depth of  $\varphi$ .]

By the definition of “ $M$  is  $\Lambda$ -generic” and categoricity of  $K_\mu^*$  we are done.  $\square_{2.11}$

**2.12 Conclusion.** If  $\mu \geq (2^{<\theta})^+, \theta > \text{LS}(\bar{\mathfrak{K}})$  and  $\text{cf}(\mu) \geq \theta > \text{LS}(\bar{\mathfrak{K}})$  then every  $M \in K_\mu^*$  is  $\mathbb{L}_{\infty, \theta}[\bar{\mathfrak{K}}]$ -generic, hence if  $M_1 \leq_{\bar{\mathfrak{K}}} M_2$  are from  $K_\mu^*$  then  $M_1 \prec_{\mathbb{L}_{\infty, \theta}[\bar{\mathfrak{K}}]} M_2$ .

*Remark.* 1) With a little more care, if  $\mu = \mu_0^+$  also  $\theta = \mu$  is O.K. but here this is prepheral.

2)  $\theta \leq \text{LS}(\bar{\mathfrak{K}})$  is not problematic, we just ignore it.

3) So 2.12 improve 1.12, i.e. we need  $\text{cf}(\mu) \geq \lambda(> \text{LS}(\bar{\mathfrak{K}}))$  instead  $\mu = \mu^{<\lambda}$  but still there is a class of  $\mu$  which are not covered.

*Proof.* Let  $\Lambda_*$  be as in 2.8(2) so in particular  $|\Lambda_*| \leq 2^{<\theta}$ . Now 2.11(6) and clause (g) of 2.8 proves the first assertion in 2.12. For the second assume that  $M_1 \leq_{\bar{\mathfrak{K}}} M_2$  and we shall prove that  $M_1 \prec_{\mathbb{L}_{\infty, \theta}[\bar{\mathfrak{K}}]} M_2$ .

By the categoricity of  $\mathfrak{K}$  in  $\mu$  or clause  $(b)_\mu^-$  of Hypothesis 2.1,  $K^*$  is categorical in  $\mu$  hence  $M_1, M_2 \in K_\mu^*$  are  $\Lambda_*$ -generic. Suppose  $\bar{a} \in {}^{\ell g(\bar{x})}(M_1), \varphi(\bar{x}) \in \Lambda_*$ , so by  $M_1'$  being  $\Lambda_*$ -generic (or  $\boxtimes$  from the end of the proof of 2.11 applied to  $\bar{M}^2$ ) we have

$$(*)_1 \quad M_1 \models \varphi[\bar{a}] \Rightarrow M_1 \Vdash \varphi[\bar{a}] \Rightarrow M_1 \models \varphi[\bar{a}]$$

and by  $M_2$  being  $\Lambda_*$ -generic (or  $\boxtimes$  from the end of the proof of 2.11 applied to  $\bar{M}^2$ ) we have

$$(*)_2 \quad M_2 \models \varphi[\bar{a}] \Rightarrow M_2 \Vdash \varphi[\bar{a}] \Rightarrow M_2 \models \varphi[\bar{a}]$$

and by the definition of “ $M \Vdash \varphi[\bar{a}]$ ” recalling  $M_1 \leq_{\mathfrak{K}_\mu} M_2$ ,

$$(*)_3 \quad \text{if } M_1 \Vdash \varphi'[\bar{a}] \text{ then } M_2 \Vdash \varphi'[\bar{a}] \text{ for } \varphi'(\bar{x}) \in \{\varphi(\bar{x}), \neg\varphi(\bar{x})\}.$$

So both  $M_1$  and  $M_2$  satisfy  $\varphi[\bar{a}]$  if  $M_1$  satisfy it, but this applies to  $\neg\varphi[\bar{a}]$  too; so we are done.  $\square_{2.12}$

**2.13 Claim.** *If  $K$  is categorical also in  $\mu^*$  or just Hypothesis 2.7 apply also to  $\mu^*$ , too, (with the same  $\Phi$ ) and  $\mu^* \geq \mu^{<\theta} > \mu > \theta > \text{LS}(\mathfrak{K})$  and  $(*)$  below, then every  $M \in K_\mu^*$  is  $\mathbb{L}_{\infty, \theta}[\mathfrak{K}]$ -generic and  $M_1 \in K_\mu^* \wedge M_2 \in K_\mu^* \wedge M_1 \leq_{\mathfrak{K}_\mu} M_2 \Rightarrow M_1 \prec_{\mathbb{L}_{\infty, \theta}[\mathfrak{K}]} M_2$ , i.e. the conclusions of 1.12, 2.12 hold where*

$$(*) \quad \text{if } M \in K_{\mu^*}^* \text{ and } A \in [M]^\mu \text{ then we can find } N \leq_{\mathfrak{K}} M \text{ such that } A \subseteq N \in K_\mu^* \\ \text{and for every } \varphi(\bar{x}) \in \mathbb{L}_{\infty, \theta}[\mathfrak{K}] \text{ and } \bar{a} \in {}^{\ell g(\bar{x})}N \text{ we have } M \Vdash \varphi[\bar{a}] \Leftrightarrow N \Vdash \varphi[\bar{a}].$$

*Proof.* We shall choose  $(M_i, N_i) \in K_{\mu^*, \mu}$  by induction on  $i \leq \theta^+$  such that not only  $M_i \in K_{\mu^*}^*$  (see the definition of  $K_{\mu^*, \mu}$ ) but also  $N_i \in K_\mu^*$  and this sequence of pairs is  $\leq_{\mathfrak{K}}$ -increasing continuous. For  $i = 0$  use any pair, e.g.  $M_0 = \text{EM}_{\tau(\mathfrak{K})}(\mu^*, \Phi)$  and  $N_0 = \text{EM}_{\tau(\mathfrak{K})}(\mu, \Phi)$ .

For  $i$  limit take unions, recalling  $M_j, N_j$  are pseudo superlimit for  $j < i$ .

For  $i = j + 1$ , let  $N_j^+ \leq_{\mathfrak{K}} M_j$  be such that  $N_j \subseteq N_j^+ \in K_\mu$  and  $(M_j, N_j^+)$  satisfies  $(*)$  of the claim (standing for  $(M, N)$ ). Let  $\Lambda_*$  be as in 2.8 for  $\mu^*$ . Then by 2.11(5) with  $(\mu^*, \mu, \theta)$  here standing for  $(\mu, \chi, \theta)$  there noting that in  $\square(c)$  there we use the case  $\chi^{<\theta} \leq \mu$  which here means  $\mu = \mu^{<\theta}$ , we can choose a  $\Lambda_*$ -generic pair  $(M_i, N_i) \in K_{\mu^*, \mu}$  above  $(M_j, N_j^+)$  hence by 2.8(2)(g) also it is a  $\mathbb{L}_{\infty, \theta}[\mathfrak{K}]$ -generic pair. Now for  $j < \theta^+$ , for  $\bar{a} \in {}^{\theta^>}(N_j)$ , we can read  $\text{gtp}_\theta^{\mu^*}(\bar{a}, \emptyset, M_{j+1})$  and it is complete, but as by our use of  $(*)$  it is the same as  $\text{gtp}_\theta^{\mu}(\bar{a}, \emptyset, N_{j+1}^+)$ .

So  $\text{gtp}_\theta^\mu(\bar{a}, \emptyset, N_{j+1}^+)$  is complete for every  $\bar{a} \in {}^{\theta>}(N_j)$ , so also  $\text{gtp}^\mu(\bar{a}, \emptyset, N_{\theta+})$  is complete by monotonicity.

Now if  $\bar{a} \in {}^{\theta>}(N_{\theta+})$  then for some  $j < \theta^+$  we have  $\bar{a} \in {}^{\theta>}(N_j)$ , so by the above  $p_{\bar{a}} := \text{gtp}_\theta^{\mu^*}(\bar{a}, \emptyset, M_{j+1}) = \text{gtp}_\theta^\mu(\bar{a}, \emptyset, N_{j+1}^+) = \text{gtp}_\theta^\mu(\bar{a}, \emptyset, N_{\theta+})$  is complete and does not depend on  $j$  as long as  $j$  is large enough.

Now we prove that if  $\bar{a} \in {}^{\theta>}(N_{\theta+})$  then  $\varphi(\bar{x}) \in p_{\bar{a}} \Rightarrow N_{\theta+} \models \varphi[\bar{a}]$ ; and we prove this by induction on the quantifier depth of  $\varphi(\bar{x})$ ; as usual the real case is  $\varphi(\bar{x}) = (\exists \bar{y})\varphi(\bar{y}, \bar{x})$ . Let  $j < \theta^+$  be such that  $\bar{a} \in {}^{\ell g(\bar{x})}(N_j)$ , so  $p_{\bar{a}} = \text{gtp}_\theta^{\mu^*}(\bar{a}, M_{j+1})$  so  $M_{j+1} \models \varphi[\bar{a}]$  and by the choice of  $(M_{j+1}, N_{j+1})$  it follows that  $N_{j+1} \models \varphi[\bar{a}]$  hence for some  $\bar{b} \in {}^{\ell g(\bar{y})}(N_{j+1})$  we have  $N_{j+1} \models \psi[\bar{b}, \bar{a}]$  hence  $M_{j+1} \models \psi(\bar{b}, \bar{a})$ , hence  $\psi(\bar{y}, \bar{x}) \in p_{\bar{b} \wedge \bar{a}}$  hence by the induction hypothesis  $N_{\theta+} \models \psi[\bar{b}, \bar{a}]$  hence  $N_{\theta+} \models \varphi[\bar{a}]$ .  $\square_{2.13}$

**2.14 Conclusion.** 1) For each  $\theta \geq \text{LS}(\mathfrak{K})$  the family of  $\mu > 2^{<\theta}$  in which  $K$  is categorical but some (equivalent every)  $M \in K_\mu$  is not  $\mathbb{L}_{\infty, \theta}[\mathfrak{K}]$ -generic is  $\subseteq \{[\mu_i, \mu_i^{<\theta}] : i < 2^{2^\theta}\}$  for some sequence  $\langle \mu_i : i < 2^{2^\theta} \rangle$  of cardinals.

2) Similarly for pseudo solvable, i.e. for each  $\theta \geq \text{LS}(\mathfrak{K})$  and  $\Phi \in \Upsilon_\theta^{\text{or}}$  for at most  $\beth_2(\theta)$  cardinals  $\mu > 2^{<\theta}$  we have  $(\forall \alpha < \mu)(|\alpha|^{<\theta} < \mu)$  and for some  $\mu^* \in [\mu, \mu^{<\theta}]$  the pair  $(\mathfrak{K}, \Phi)$  is pseudo  $\mu^*$ -solvable but some  $\equiv$  every  $M \in K_{\Phi, \mu^*}^*$  is not  $\mathbb{L}_{\infty, \theta+}[\mathfrak{K}]$ -generic.

*Proof.* Straight. Note that it is enough to prove this for each  $\Phi$  separately.

Toward contradiction assume  $\langle \mu_\varepsilon : \varepsilon < (\beth_2(\theta))^+ \rangle$  is an increasing sequence of such cardinals, satisfying  $(\mu_\varepsilon)^{<\theta} < \mu_{\varepsilon+1}$  and choose  $I_\varepsilon \times \mu_\varepsilon \times (2^{<\theta})^+$ , hence  $\langle I_\varepsilon : \varepsilon < (\beth_2(\theta))^+ \rangle$  is an increasing sequence of linear orders as in 2.9, in particular, well ordered. Let  $\Lambda_\varepsilon = \Lambda_{\mathfrak{K}, \Phi, \mu_\varepsilon, \theta}^*$  be from 2.8(2) applied to  $I = I_\varepsilon$  hence to any ordinal  $< \mu_\varepsilon^+$  of cofinality  $> 2^{<\theta}$ . Now the number of functions  $\mathbf{t}_{\mathfrak{K}, \Lambda_\varepsilon}^{\mu_\varepsilon}$  (see Observation 2.9(3) and Definition 2.10(2)) is at most  $\beth_2(\theta)$ , so for some  $\varepsilon < \zeta < (\beth_2(\theta))^+$  we have  $\mathbf{t}_{\mathfrak{K}, \Lambda_\varepsilon}^{\mu_\varepsilon} = \mathbf{t}_{\mathfrak{K}, \Lambda_\zeta}^{\mu_\zeta}$ .

Now apply 2.13 with  $(\mu_\zeta, \mu_\varepsilon)$  here standing for  $(\mu^*, \mu)$  there,  $(*)$  there holds easily by 2.9(3) so we get a contradiction.  $\square_{2.14}$

\* \* \*

For the rest of this section we note some basic facts on the dependency on  $\Phi$  (not used here).

**2.15 Definition.** 1) We define a two-place relation  $\mathcal{E}_\kappa = \mathcal{E}_\kappa^{\text{or}}[\mathfrak{K}]$  on  $\Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$ , so  $\kappa \geq \text{LS}(\mathfrak{K}) : \Phi_1 \mathcal{E}_\kappa \Phi_2$  iff for every linear orders  $I_1, I_2$  there are linear orders  $J_1, J_2$  extending  $I_1, I_2$  respectively such that  $\text{EM}_{\tau(\mathfrak{K})}(J_1, \Phi_1), \text{EM}_{\tau(\mathfrak{K})}(J_2, \Phi_2)$  are isomorphic. 2) We define  $\leq_\kappa^{\text{or}} = \leq_\kappa^{\text{or}}[\mathfrak{K}]$ , a two-place relation on  $\Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$  as in part (1) only in the end  $\text{EM}_{\tau(\mathfrak{K})}(J_1, \Phi_1)$  can be  $\leq_{\mathfrak{K}}$ -embedded into  $\text{EM}_{\tau(\mathfrak{K})}(J_2, \Phi_2)$ .

**2.16 Claim.** 1) The following conditions on  $\Phi_1, \Phi_2 \in \Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$  are equivalent

- (a)  $\Phi_1 \mathcal{E}_\kappa \Phi_2$
- (b) there are  $I_1, I_2 \in K^{\text{lin}}$  of cardinality  $\geq \beth_{1,1}(\kappa)$  such that  $\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi_1), \text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi_2)$  are isomorphic
- (c) there are  $\Phi'_1, \Phi'_2$  satisfying  $\Phi_\ell \leq^\otimes \Phi'_\ell \in \Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$  for  $\ell = 1, 2$  such that  $\Phi'_1, \Phi'_2$  are essentially equal (see Definition 2.17 below).

2) The following conditions are equivalent

- (a)  $\Phi_1 \leq_\kappa^{\text{or}} \Phi_2$  recall  $\leq_\kappa = \leq_\kappa^{\text{or}}[\mathfrak{K}]$
- (b) there are  $I_1, I_2 \in K^{\text{lin}}$  of cardinality  $\geq \beth_{1,1}(\kappa)$  such that  $\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi_1)$  can be  $\leq_{\mathfrak{K}}$ -embedded into  $\text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi_2)$
- (c) for every  $I_1 \in K^{\text{lin}}$  there is  $I_2 \in K^{\text{lin}}$  such that  $\text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi_1)$  can be  $\leq_{\mathfrak{K}}$ -embedded into  $\text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi_2)$ .

**2.17 Definition.**  $\Phi_1, \Phi_2 \in \Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$  are essentially equal when for every linear order  $I$  there is an isomorphism  $f$  from  $\text{EM}_{\tau(\mathfrak{K})}(I, \Phi_1)$  onto  $\text{EM}_{\tau(\mathfrak{K})}(I, \Phi_2)$  such that for any  $\tau_{\Phi_1}$ -term  $\sigma_1(x_0, \dots, x_{n-1})$  there is a  $\tau_{\Phi_2}$ -term  $\sigma_2(x_0, \dots, x_{n-1})$  such that:  $t_0 <_I \dots <_I t_{n-1} \Rightarrow f(a_1) = a_2$ , where  $a_\ell$  is  $\sigma_\ell(a_{t_0}, \dots, a_{t_{n-1}})$  as computed in  $\text{EM}(I, \Phi_\ell)$  for  $\ell = 1, 2$ .

*Proof of 2.16.* Straight (particularly recalling such proof in 1.29(1)). □<sub>2.17</sub>

**2.18 Claim.** 1)  $\mathcal{E}_\kappa = \mathcal{E}_\kappa^{\text{or}}[\mathfrak{K}]$  is an equivalence relation and  $\Phi_1 \mathcal{E}_\kappa^{\text{or}}[\mathfrak{K}] \Phi_2 \Rightarrow \Phi_1 \leq_\kappa^{\text{or}}[\mathfrak{K}] \Phi_2$ .

1A) In fact if  $\langle \Phi_\varepsilon : \varepsilon < \varepsilon(*) \rangle$  are pairwise  $\mathcal{E}_\kappa$ -equivalent and  $\varepsilon(*) \leq \kappa$  then we can find  $\langle \Phi'_\varepsilon : \varepsilon < \kappa \rangle$  satisfying  $\Phi'_\varepsilon \leq^\otimes \Phi'_\varepsilon$  for  $\varepsilon < \varepsilon(*)$  such that the  $\Phi'_\varepsilon$  for  $\varepsilon < \varepsilon(*)$  are pairwise essentially equal.

2)  $\leq_\kappa^{\text{or}}$  is a partial order.

3) If  $\Phi_1, \Phi_2 \in \Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$  are essentially equal then  $(\mathfrak{K}, \Phi_1)$  is pseudo/weakly/strongly  $(\mu, \kappa)$ -solvable iff  $(\mathfrak{K}, \Phi_2)$  is pseudo/weakly/strongly  $(\mu, \kappa)$ -solvable.



- 4) If  $\Phi_1 \in \Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$  is strongly  $(\mu, \kappa)$ -solvable and  $\Phi_2$  exemplifies  $\mathfrak{K}$  is  $(\mu, \kappa)$ -solvable then  $\Phi_1 \mathcal{E}_\kappa \Phi_2$ .
- 5) If  $\mathfrak{K}$  is categorical in  $\mu$  and  $\mu > \kappa \geq \text{LS}(\mathfrak{K})$  then every  $\Phi \in \Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$  is strongly  $(\mu, \kappa)$ -solvable.
- 6) Assume  $(\mathfrak{K}, \Phi_\ell)$  is pseudo  $(\mu, \kappa)$ -solvable and  $\mu \geq \beth_{1,1}(\kappa)$  for  $\ell = 1, 2$ . Then  $\Phi_1 \mathcal{E}_\kappa \Phi_2$  iff  $\Phi_1 \leq_\kappa^{\text{or}} [\mathfrak{K}] \Phi_2 \wedge \Phi_2 \leq_\kappa^{\text{or}} [\mathfrak{K}] \Phi_1$ .
- 7) If  $\Phi_1 \leq_\kappa^{\text{or}} \Phi_2$  and  $\Phi_1$  is strongly  $(\mu, \kappa)$ -solvable or just pseudo  $(\mu, \kappa)$ -solvable then  $\Phi_1, \Phi_2$  are  $\mathcal{E}_\kappa^{\text{or}}[\mathfrak{K}]$ -equivalent.

*Proof.* Easy, use 1.29(1) and its proof. □<sub>2.18</sub>

### §3 CATEGORICITY FOR CARDINALS ON A CLUB

We draw here an easy conclusion from §2, getting that on a closed unbounded class of cardinals which is  $\aleph_0$ -closed we get a constant answer to being categorical. This is, of course, considerably weaker than conjecture 0.1 but still is a progress, e.g. it shows that the categoricity spectrum is not totally chaotic.

We concentrate on the case the results of §1 holds (e.g.  $\mu = \mu^\lambda$ ) for the  $\lambda$ 's with which we deal. To eliminate this extra assumption we need §2. This section is not used later. Note that 3.4 is continued (and improved) in [Sh:F820] and Exercise 3.8, [Sh:F782] improve 3.6; similarly 3.7.

In the claims below we concentrate on fix points of the sequence of  $\beth_\alpha$ 's.

*3.1 Hypothesis.* As in Hypothesis 1.2, (i.e.  $\mathfrak{K}$  is an a.e.c. with models of arbitrarily large cardinality).

**3.2 Definition.** 1) Let  $\text{Cat}_{\mathfrak{K}}$  be the class of cardinals in which  $\mathfrak{K}$  is categorical.

1A) Let  $\text{Sol} = \text{Sol}_{\mathfrak{K}, \Phi} = \text{Sol}_{\mathfrak{K}, \Phi}^1$  be the class of  $\mu > \text{LS}[\mathfrak{K}]$  such that  $(\mathfrak{K}, \Phi)$  is pseudo  $\mu$ -solvable. Let  $\text{Sol}_{\mathfrak{K}, \Phi}^2[\text{Sol}_{\mathfrak{K}, \Phi}^3]$  be the class of  $\mu > \text{LS}(\mathfrak{K})$  such that  $(\mathfrak{K}, \Phi)$  is weakly [strongly]  $\mu$ -solvable.

2) Let  $\text{mod-com}_{\mathfrak{K}, \Phi}$  be the class of pairs  $(\mu, \theta)$  such that:  $\mu > \theta \geq \text{LS}(\mathfrak{K})$  and  $\mathbb{L}_{\infty, \theta+}[\mathfrak{K}]$  is  $\mu$ -model complete (on  $K_{\Phi, \mu}^*$ , see Definition 2.3(3)(b), 2.3(5)).

3) Let  $\text{Cat}'_{\mathfrak{K}}$  be the class of  $\mu \in \text{Cat}_{\mathfrak{K}}$  such that:  $\mu \geq \beth_{1,1}(\text{LS}(\mathfrak{K}))$  and if  $\text{LS}(\mathfrak{K}) \leq \theta$  and  $\beth_{1,1}(\theta) \leq \mu$  then  $\mathbb{L}_{\infty, \theta+}[\mathfrak{K}]$  is  $\mu$ -model complete.

3A) For  $\Phi \in \Upsilon_{\mathfrak{K}}^{\text{or}}$  let  $\text{Sol}_{\mathfrak{K}, \Phi}^{k,*}$  be the class of  $\mu \in \text{Sol}_{\mathfrak{K}, \Phi}^k$  such that  $\mu \geq \beth_{1,1}(\text{LS}(\mathfrak{K}))$  and: if  $\text{LS}(\mathfrak{K}) \leq \theta$  and  $\beth_{1,1}(\theta) \leq \mu$  then the pair  $(\mathbb{L}_{\infty, \theta+}[\mathfrak{K}], \Phi)$  is  $\mu$ -model complete.

Let  $\text{Sol}_{\mathfrak{K}, \Phi}^{\ell, < \theta}$  be the class of  $\lambda \in \text{Sol}_{\mathfrak{K}, \Phi}^\ell$  such that  $\mathbb{L}_{\infty, \theta}[\mathfrak{K}]$  is  $\mu$ -model complete (see V§2).

Let  $\text{Sol}'_{\mathfrak{K}, \Phi} = \text{Sol}_{\mathfrak{K}, \Phi}^{1,*}$ . Instead  $k, *$  we may write  $3 + k$ .

4) Let  $\mathbf{C} = \{\lambda : \lambda = \beth_\lambda \text{ and } \text{cf}(\lambda) = \aleph_0\}$ .

**3.3 Exercise:** 1) The conclusion of 1.12(1) equivalently 1.12(2) means that  $\theta \leq \lambda \Rightarrow (\mu, \theta) \in \text{mod-com}_{\mathfrak{K}, \Phi}$ .

2) Write down the obvious implications.

**3.4 Claim.** *If  $\mu > \lambda = \beth_\lambda > \kappa \geq \text{LS}(\mathfrak{K})$  and  $\Phi \in \Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$ ,  $\text{cf}(\lambda) = \aleph_0$  then  $\mu = \mu^{<\lambda} \Rightarrow \mu \in \text{Sol}'_{\mathfrak{K}, \Phi} \Rightarrow \lambda \in \text{Sol}'_{\mathfrak{K}, \Phi}$ .*

*Proof.* The first implication holds by 1.12(2) and 3.3. The second implication, its assumption implies Hypothesis 1.18, see 3.3(1) hence its conclusion holds by 1.41.

□<sub>3.4</sub>

**3.5 Observation.**  $K_\lambda$  is categorical in  $\lambda$  (hence Hypothesis 1.18 holds), if:

$$\circledast_\lambda \quad \lambda = \beth_\lambda = \sup(\lambda \cap \text{Cat}'_{\mathfrak{K}}) > \text{LS}(\mathfrak{K}) \text{ and } \aleph_0 = \text{cf}(\lambda).$$

*Proof.* Fix  $\Phi \in \Upsilon_{\aleph_0}^{\text{or}}$ , now clearly  $\text{Sol}'_{\mathfrak{K}, \Phi} \supseteq \text{Cat}'_{\mathfrak{K}}$  by their definitions.

By the assumptions we can find  $\langle \mu_n : n < \omega \rangle$  such that  $\lambda = \Sigma\{\mu_n : n < \omega\}$ ,  $\text{LS}(\mathfrak{K}) < \mu_n \in \text{Cat}'_{\mathfrak{K}}$  and  $\beth_{1,1}(\mu'_n) < \mu_{n+1}$  where  $\mu'_n = \beth_{1,1}(\mu_n)$ . As every  $M \in K_{\mu_{n+1}}$  is  $\mathbb{L}_{\infty, \mu'_n}[\mathfrak{K}]$ -generic (as  $K_{\mu_{n+1}} \subseteq K_{\Phi, \mu_{n+1}}$  and  $\mu_{n+1} \in \text{Cat}'_{\mathfrak{K}}$ ) easily

$$(*)_0 \quad \text{if } M \leq_{\mathfrak{K}} N \text{ are from } K_{\Phi, \geq \mu_{n+1}}^* \text{ then } M \prec_{\mathbb{L}_{\infty, \mu'_n}[\mathfrak{K}]} N.$$

Let  $M^\ell \in K_\lambda$ , for  $\ell \in \{1, 2\}$ ; so we can find a  $\leq_{\mathfrak{K}}$ -increasing sequence  $\langle M_n^\ell : n < \omega \rangle$  such that  $M_n^\ell \in K_{\mu_n}$ ,  $M_n^\ell \leq_{\mathfrak{K}} M_{n+1}^\ell \leq_{\mathfrak{K}} M^\ell$  and  $M^\ell = \cup\{M_n^\ell : n < \omega\}$ . Now

$$(*)_1 \quad M_n^\ell \in K_{\Phi, \mu_n}^*.$$

[Why? As  $\mathfrak{K}$  is categorical in  $\mu_n = \|M_n^\ell\|$ .]

$$(*)_2 \quad \text{if } \alpha \leq \mu_n, n < m < k \text{ and } \bar{a}, \bar{b} \in {}^\alpha(M_m^\ell) \text{ then:}$$

$$(a) \quad \text{tp}_{\mathbb{L}_{\infty, \mu'_n}[\mathfrak{K}]}(\bar{a}, \emptyset, M_m^\ell) = \text{tp}_{\mathbb{L}_{\infty, \mu'_n}}(\bar{b}, \emptyset, M_m^\ell) \text{ iff } \text{tp}_{\mathbb{L}_{\infty, \mu'_n}[\mathfrak{K}]}(\bar{a}, \emptyset, M_k^\ell) = \text{tp}_{\mathbb{L}_{\infty, \mu'_n}}(\bar{b}, \emptyset, M_k^\ell).$$

$$(b) \quad \text{if } \text{tp}_{\mathbb{L}_{\infty, \mu'_n}[\mathfrak{K}]}(\bar{a}, \emptyset, M_k^\ell) = \text{tp}_{\mathbb{L}_{\infty, \mu'_n}[\mathfrak{K}]}(\bar{b}, \emptyset, M_k^\ell) \text{ then } \text{tp}_{\mathbb{L}_{\infty, \mu'_m}[\mathfrak{K}]}(\bar{a}, \emptyset, M_k^\ell) = \text{tp}_{\mathbb{L}_{\infty, \mu'_m}}(\bar{b}, \emptyset, M_k^\ell).$$

[Why? Clause (a) by  $(*)_0$ , clause (b) by 1.19(3).]

$$(*)_3 \quad M_n^1 \cong M_n^2.$$

[Why? As  $\mathfrak{K}$  is categorical in  $\mu_n$ .]

We now proceed as in the proof of 1.38. Let  $\mathcal{F}_n = \{f: \text{for some } \bar{a}_1, \bar{a}_2 \text{ and } \alpha < \mu_n \text{ we have } \bar{a}_\ell \in {}^\alpha(M_{n+2}^\ell) \text{ for } \ell = 1, 2, \text{tp}_{\mathbb{L}_{\infty, \mu_{n+1}[\mathfrak{K}]}(\bar{a}_1, \emptyset, M_{n+2}^1) = \text{tp}_{\mathbb{L}_{\infty, \mu_{n+1}[\mathfrak{K}]}(\bar{a}_2, \emptyset, M_{n+2}^2)} \text{ and } f \text{ is the function which maps } \bar{a}_1 \text{ into } \bar{a}_2\}$ , (actually can use  $\alpha = \mu_n$ ).

By the hence and forth argument we can find  $f_n \in \mathcal{F}_n$  by induction on  $n < \omega$  such that  $M_n^1 \subseteq \text{Dom}(f_{2n+2})$ ,  $M_n^2 \subseteq \text{Rang}(f_{2n+2})$  and  $f_n \subseteq f_{n+1}$ ; hence  $\cup\{f_n : n < \omega\}$  is an isomorphism from  $M^1$  onto  $M^1$ .  $\square_{3.4}$

**3.6 Claim.**  $\mathfrak{K}$  is categorical in  $\lambda$  when:

$$\circledast_\lambda^+ \quad \lambda = \beth_\lambda > \text{LS}(\mathfrak{K}) \text{ and } \lambda = \text{otp}(\text{Cat}_{\mathfrak{K}} \cap \lambda \cap \mathbf{C}) \text{ and } \text{cf}(\lambda) = \aleph_0.$$

*Proof.* Fix  $\Phi$  as in the proof of 3.4. Let  $\langle \theta_n : n < \omega \rangle$  be increasing such that  $\lambda = \Sigma\{\theta_n : n < \omega\}$  and  $\text{LS}(\mathfrak{K}) < \theta_0$ . For each  $n$ , by 2.14 we know  $\{\mu \in \text{Cat}_{\mathfrak{K}} : \mu > \theta_n \text{ and the } M \in K_\mu \text{ is not } \mathbb{L}_{\infty, \theta_n^+}\text{-generic}\}$  is “not too large”, i.e. is included in the union of at most  $\beth_2(\theta_n)$  intervals of the form  $[\chi, \chi^{\theta_n}]$ . Now we choose  $(n(\ell), \mu_\ell)$  by induction on  $\ell < \omega$  such that

- $\circledast \quad (a) \quad n(\ell) < \omega \text{ and } \mu_\ell \in \text{Cat}_{\mathfrak{K}} \cap \lambda$
- $(b) \quad \text{if } \ell = k + 1 \text{ then } n(\ell) > n(k), \theta_{n(\ell)} > \mu_k, \mu_\ell \in \text{Cat}_{\mathfrak{K}} \cap \lambda \setminus \theta_{n(\ell)}^+ \text{ and the } M \in K_{\mu_\ell} \text{ is } \mathbb{L}_{\infty, \theta_{n(\ell)}}[\mathfrak{K}]\text{-generic (hence } \mathbb{L}_{\infty, \mu_k^+}[\mathfrak{K}]\text{-generic)}.$

This is easy and then continue as in 3.5.  $\square_{3.6}$

We have essentially proved

**3.7 Theorem.** In 3.5, 3.6 we can use  $\text{Sol}_{\mathfrak{K}, \Phi}$ ,  $\text{Sol}'_{\mathfrak{K}, \Phi}$  instead of  $\text{Cat}_{\mathfrak{K}}$ ,  $\text{Cat}'_{\mathfrak{K}}$ .

**3.8 Exercise:** For Claim 1.38(2), Hypothesis 1.18 suffice.

[Hint: The proof is similar to the existing one using 1.19.]

## §4 GOOD FRAMES

Here comes the main result of Chapter V: from categoricity (or solvability) assumptions we derive the existence of good  $\lambda$ -frames.

Our assumption is such that we can apply §1.

## 4.1 Hypothesis. 1)

- (a)  $\mathfrak{K}$  is an a.e.c.
- (b)  $\mu > \lambda = \beth_\lambda > \text{LS}(\mathfrak{K})$  and  $\text{cf}(\lambda) = \aleph_0$ ;
- (c)  $\Phi \in \Upsilon_{\mathfrak{K}}^{\text{or}}$
- (d)  $\mathfrak{K}$  is categorical in  $\mu$  or just
- (d)<sup>-</sup>  $(\mathfrak{K}, \Phi)$  is pseudo superlimit in  $\mu$  (this means  $\Phi \in \text{Sol}_{\mathfrak{K}, \Phi}^1$ ; so 1.18(1) holds)
- (e) also 1.18(2)(a) holds, i.e. the conclusion of 1.12(2) holds.

2) In addition we may use some of the following but then we mention them and (we add superscript  $*$  when used; note that  $(g) \Rightarrow (f)$  by 1.39)

- (f)  $K_\lambda^*$  is closed under  $\leq_{\mathfrak{K}}$ -increasing unions (justified by 1.38)
- (g)  $\langle \lambda_n : n < \omega \rangle$  is increasing,  $\lambda_0 > \text{LS}(\mathfrak{K})$ ,  $\lambda = \Sigma\{\lambda_n : n < \omega\}$  and the assumptions of 1.38 holds.

4.2 Observation. 1)  $\mathfrak{K}_\lambda^*$  is categorical.

2)  $\mathfrak{K}_\lambda^*$  has amalgamation.

3)\* (We assume (f) of 4.1(2)).  $\mathfrak{K}_\lambda$  is a  $\lambda$ -a.e.c.

*Proof.* 1) By 1.16(1) or 1.19(4) as  $\text{cf}(\lambda) = \aleph_0$ .

2) By 1.30(1).

3) As in 1.39, (i.e. as  $\leq_{\mathfrak{K}_\lambda^*} = \leq_{\mathfrak{K}} \upharpoonright \mathfrak{K}$ , closure under unions of  $\leq_{\mathfrak{K}}$ -increasing chains is the only problematic point and it holds by (f) of 4.1(2)).  $\square_{4.2}$

4.3 Remark. 1) Why do we not assume 4.1(1),(2) all the time? The main reason is that for proving some of the results assuming 4.1(1),(2) we use some such results on smaller cardinals on which we use 4.1(1) only.

2) Note that it is not clear whether improvement by using 4.1(1) only will have any affect when (or should we say if) we succeed to have the parallel of IV§12.

- 4.4 Claim.** 1) Assume  $M_0 \leq_{\mathfrak{K}_\lambda^*} M_\ell$ ,  $\alpha < \lambda$  and  $\bar{a}_\ell \in {}^\alpha(M_\ell)$  for  $\ell = 1, 2$  and  $\kappa := \beth_{1,1}(\beth_2(\theta)^+)$  where  $\theta := |\alpha| + \text{LS}(\mathfrak{K})$  so  $\kappa < \lambda$ . If  $\text{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar{a}_1, M_0, M_1) = \text{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar{a}_2, M_0, M_2)$  then  $\text{tp}_{\mathfrak{K}_\lambda^*}(\bar{a}_1, M_0, M_1) = \text{tp}_{\mathfrak{K}_\lambda^*}(\bar{a}_2, M_0, M_2)$ .
- 2) If  $M_1 \leq_{\mathfrak{K}_\lambda^*} M_2$  then  $M_1 \prec_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} M_2$  for every  $\theta < \lambda$ , and moreover  $M_1 \prec_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]} M_2$ .
- 2A) If  $M_0 \leq_{\mathfrak{K}_\lambda^*} M_\ell$  for  $\ell = 1, 2$  and  $\text{tp}_{\mathfrak{K}_\lambda^*}(\bar{a}_1, M_0, M_1) = \text{tp}_{\mathfrak{K}_\lambda^*}(\bar{a}_2, M_0, M_2)$  and  $\bar{a}_\ell \in {}^\alpha(M_0)$ ,  $\alpha < \kappa \leq \lambda$  then  $\text{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar{a}_1), M_0, M_1) = \text{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar{a}_2, M_0, M_2)$ .
- 2B) In part (1), if  $M_\ell \leq_{\mathfrak{K}_\lambda^*} M'_\ell$  for  $\ell = 1, 2$  then  $\text{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar{a}_1, M, M'_1) = \text{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar{a}_2, M, M'_2)$ .
- 3) Assume that  $M_0 \leq_{\mathfrak{K}_\lambda^*} M_1 \leq_{\mathfrak{K}_\lambda^*} M_2 \leq_{\mathfrak{K}_\lambda^*} M_3$ ,  $\bar{a} \in {}^\alpha(M_2)$ ,  $\alpha < \lambda$  and  $\kappa = \beth_{1,1}(|\alpha| + \text{LS}(\mathfrak{K})) < \theta < \lambda$ . Then
- (a) from  $\text{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$  we can compute  $\text{tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$  and  $\text{tp}_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]}(\bar{a}, M_0, M_3)$
  - (b) from  $\text{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar{a}, \emptyset, M_2)$  we can compute  $\text{tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar{a}, \emptyset, M_2)$  and even  $\text{tp}_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]}(\bar{a}, \emptyset, M_2)$
  - (c) from  $\text{tp}_{\mathfrak{K}_\lambda^*}(\bar{a}, M_1, M_2)$  we can compute  $\text{tp}_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$  and  $\text{tp}_{\mathfrak{K}_\lambda^*}(\bar{a}, M_0, M_3)$ .
- 4) If  $M_1 \leq_{\mathfrak{K}_\lambda^*} M_2$  and  $\alpha < \kappa^* < \lambda$ ,  $\mathbf{I}_\ell \subseteq {}^\alpha(M_1)$ ,  $|\mathbf{I}_\ell| > \kappa$ ,  $\mathbf{I}_\ell$  is  $(\mathbb{L}_{\infty,\theta}[\mathfrak{K}], \kappa^*)$ -convergent in  $M_1$  for  $\ell = 1, 2$  and  $\text{Av}_{<\kappa}(\mathbf{I}_1, M_1) = \text{Av}_{<\kappa}(\mathbf{I}_1, M_1)$  then  $\mathbf{I}_\ell$  is  $(\mathbb{L}_{\infty,\kappa}[\mathfrak{K}], \kappa^*)$ -convergent in  $M_\ell$  for  $\ell = 1, 2$  and  $\text{Av}_{<\kappa}(\mathbf{I}_1, M_\ell) = \text{Av}_{<\kappa}(\mathbf{I}_1, M_2)$ .

*Proof.* 1) Without loss of generality  $M_0 = \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$  and  $I_0 \in K_\lambda^{\text{flin}}$ . By 1.29(3) for  $\ell = 1, 2$  there is a pair  $(I_\ell, f_\ell)$  such that  $I_0 \leq_{K^{\text{flin}}} I_\ell \in K_\lambda^{\text{flin}}$  and  $f_\ell$  is a  $\leq_{\mathfrak{K}}$ -embedding of  $M_\ell$  into  $M'_\ell = \text{EM}_{\tau(\mathfrak{K})}(I_\ell, \Phi)$  over  $M_0$ . By renaming without loss of generality  $f_\ell$  is the identity on  $M_\ell$  hence  $M_\ell \leq_{\mathfrak{K}} M'_\ell$ . By 1.19(1) we know that  $M_\ell \prec_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} M'_\ell$  hence  $\text{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar{a}_1, M_0, M'_1) = \text{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar{a}_1, M_0, M_1) = \text{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar{a}_2, M_0, M_2) = \text{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar{a}_2, M_0, M'_2)$ .

By 1.29(1) we can find  $(I_3, g_1, g_2, h)$  such that  $I_0 \leq_{K^{\text{flin}}} I_3 \in K_\lambda^{\text{flin}}$ ,  $g_\ell$  is a  $\leq_{\mathfrak{K}}$ -embedding of  $M'_\ell$  into  $M_4 := \text{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$  over  $M_0$  for  $\ell = 1, 2$  and  $h$  is an automorphism of  $M_4$  over  $M_0$  mapping  $g_1(\bar{a}_1)$  to  $g_2(\bar{a}_2)$ . By the definition of orbital types, this gives  $\text{tp}_{\mathfrak{K}_\lambda^*}(\bar{a}_1, M_0, M_1) = \text{tp}_{\mathfrak{K}_\lambda^*}(\bar{a}_2, M_0, M_2)$  as required.

2) This holds by 1.19(1) for  $\theta \in (\text{LS}(\mathfrak{K}), \lambda)$ , hence by 1.11(1) also for  $\theta = \lambda$  (the assumptions of 1.11 hold as clause (a) there holds by the case above  $\theta < \lambda$  and clause (b) there holds by 1.28(1)).

2A) Should be clear:

- (a) by part (2) this holds if  $\bar{a}_1 = \bar{a}_2$  and  $M_1 \leq_{\mathfrak{K}} M_2$
- (b) trivially it holds if there is an isomorphism from  $M_1$  onto  $M_2$  over  $M_0$  mapping  $\bar{a}_1$  to  $\bar{a}_2$
- (c) by the definition of  $\text{tp}$  we are done.

2B) Should be clear by part (2).

3) Clause (a):

By parts (1) + (2).

Clause (b): By 1.28(1).

Clause (c): By part (2A) and the definition of  $\mathbf{tp}$ .

4) Easy, too.  $\square_{4.4}$

**4.5 Definition.** Assume  $M_0 \leq_{\mathfrak{K}_\lambda^*} M_1 \leq_{\mathfrak{K}_\lambda^*} M_2$ ,  $\alpha < \lambda$  and  $\bar{a} \in {}^\alpha(M_2)$  and  $p = \mathbf{tp}_{\mathfrak{K}_\lambda^*}(\bar{a}, M_1, M_2)$ . We say that  $p$  does not fork over  $M_0$  (for  $\mathfrak{K}_\lambda^*$ ) when, letting  $\theta_0 = |\alpha| + \text{LS}(\mathfrak{K})$ ,  $\theta_1 = \beth_{1,1}(\beth_2(\theta_0)^+)$ ,  $\theta_2 = 2^{\theta_1}$ ,  $\theta_2 = \beth_2(\theta_1)$  we have:

- (\*) for some  $N \leq_{\mathfrak{K}^*} M_0$  satisfying  $\|N\| \leq \theta_2$  we have  $\text{tp}_{\mathbb{L}_{\infty, \theta_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$  does not split over  $N$ .

We now would like to show that there is  $\mathfrak{s}_\lambda$  which fits Chapter III and Chapter IV and  $\mathfrak{K}_{\mathfrak{s}_\lambda} = \mathfrak{K}_\lambda^*$ .

**4.6 Observation.** Assume that  $M_0 \leq_{\mathfrak{K}_\lambda^*} M_1 \leq_{\mathfrak{K}_\lambda^*} M_2$ ,  $\bar{a} \in {}^\alpha(M_2)$ ,  $\alpha < \lambda$ ,  $\lambda > \kappa_0 \geq |\alpha| + \text{LS}(\mathfrak{K})$ ,  $\kappa_1 = \beth_{1,1}(\beth_2(\kappa_0)^+)$  and  $\kappa_2 = \beth_2(\kappa_1)$ . Then the following conditions are equivalent

- (a)  $\mathbf{tp}_{\mathfrak{K}_\lambda^*}(\bar{a}, M_1, M_2)$  does not fork over  $M_0$
- (b) for some  $(\kappa_1^+, \kappa_1)$ -convergent  $\mathbf{I} \subseteq {}^\alpha(M_0)$  of cardinality  $> \kappa_2$  we have  $\text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2) = \text{Av}_{< \kappa_1}(\mathbf{I}, M_1)$  hence this type does not split over  $\cup \mathbf{I}'$  for any  $\mathbf{I}' \subseteq \mathbf{I}$  of cardinality  $> \kappa_1$
- (c) for every  $N \leq_{\mathfrak{K}} M_0$  of cardinality  $\leq \kappa_2$ , if  $\text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(\bar{a}, M_0, M_2)$  does not split over  $N$  then the type  $\text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$  does not split over  $N$ .

**4.7 Remark.** 1) See verification of axiom (E)(c) in the proof of Theorem 4.10.

2) Note that have we used  $\beth_7(\kappa_1)^+$  instead of  $\kappa_1$  in 4.5, 4.6, the difference would be small.

3) We could in clause (c) of 4.6 use “for some  $N \leq_{\mathfrak{K}} M_0$  of cardinality  $< \kappa_1$ ,  $\text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(\bar{a}, M_0, M_2)$  ...” The proof is the same.

4) We can allow below  $M_0 \leq_{\mathfrak{K}} M_1$  if  $M_0 \in K_{\geq \kappa_2}$ .

*Proof.* (a)  $\Rightarrow$  (b)

Let  $\theta_0, \theta_1, \theta_2$  be as in Definition 4.5. By Definition 4.5 there is  $N \leq_{\mathfrak{K}} M_0$  of cardinality  $\leq \theta_2$  such that

- (\*)<sub>1</sub> the type  $\text{tp}_{\mathbb{L}_{\infty, \theta_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$  does not split over  $N$ .

By Claim 1.26(1) there is a  $(\kappa_1^+, \kappa_1)$ -convergent set  $\mathbf{I} \subseteq {}^\alpha(M_0)$  of cardinality  $\kappa_2^+$  (convergence in  $M_0$ , of course) such that  $\text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(\bar{a}, M_0, M_2) = \text{Av}_{<\kappa_1}(\mathbf{I}, M_0)$ . So as  $M_0 \prec_{\mathbb{L}_{\infty, \lambda}[\mathfrak{K}]} M_1 \prec_{\mathbb{L}_{\infty, \lambda}[\mathfrak{K}]} M_2$ , by Claim 4.4(2), clearly  $\mathbf{I}$  is  $(\kappa_1^+, \kappa_1)$ -convergent also in  $M_1$  and in  $M_2$  hence  $\text{Av}_{<\kappa_1}(\mathbf{I}, M_1)$  is well defined. Hence, by Claims 1.23(2), 1.21(3) the type  $\text{Av}_{<\kappa_1}(\mathbf{I}, M_1)$  does not split over  $\cup \mathbf{I}$  but  $\theta_2 \leq \kappa_2$  and  $\cup \mathbf{I} \subseteq \cup \mathbf{I} \cup N$  hence

(\*)<sub>2</sub>  $\text{Av}_{<\theta_1}(\mathbf{I}, M_1)$  does not split over  $\cup \mathbf{I} \cup N$ .

But also

(\*)<sub>3</sub>  $\text{tp}_{\mathbb{L}_{\infty, \theta_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$  does not split over  $N$  (by the choice of  $N$ ) hence over  $\cup \mathbf{I} \cup N$ .

As  $M_0 \prec_{\mathbb{L}_{\infty, \lambda}[\mathfrak{K}]} M_1$  and  $|\cup \mathbf{I} \cup N| < \lambda$  and  $\text{tp}_{\mathbb{L}_{\infty, \theta_1}[\mathfrak{K}]}(\bar{a}, M_0, M_2) = \text{Av}_{<\theta_1}(\mathbf{I}, M_0)$  clearly, by (\*)<sub>2</sub> + (\*)<sub>3</sub> we have  $\text{tp}_{\mathbb{L}_{\infty, \theta_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2) = \text{Av}_{<\theta_1}(\mathbf{I}, M_1)$ . Now there is a pair  $(M'_2, \bar{a}')$  satisfying that  $M_1 \leq_{\mathfrak{K}} M'_2 \in K_\lambda^*$  and  $\bar{a}' \in {}^\alpha(M'_2)$  such that  $\text{tp}_{\mathbb{L}_{\infty, \theta_1}[\mathfrak{K}]}(\bar{a}', M_1, M'_2) = \text{Av}_{<\theta_1}(\mathbf{I}, M_1)$  hence by the previous sentence  $\text{tp}_{\mathbb{L}_{\infty, \theta_1}[\mathfrak{K}]}(\bar{a}', M_1, M'_2) = \text{tp}_{\mathbb{L}_{\infty, \theta_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$ . Now by 4.4(1) and then 4.4(2A) it follows that  $\text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(\bar{a}, M_1, M_0) = \text{Av}_{<\kappa_1}(\mathbf{I}, M_1)$  as required.

(b)  $\Rightarrow$  (c)

Let  $\mathbf{I}$  be as in clause (b), so  $\mathbf{I}$  is  $(\kappa_1^+, \kappa_1)$ -convergence in  $M_0$  and is of cardinality  $> \kappa_1$ . We know that  $M_0 \prec_{\mathbb{L}_{\infty, \lambda}[\mathfrak{K}]} M_1$ , so by the previous sentence,  $\mathbf{I}$  is  $(\kappa_1^+, \kappa_1)$ -convergent in  $M_1$ . To prove clause (c) assume that  $N \leq_{\mathfrak{K}} M_0$  is of cardinality  $\kappa_2$  and  $\text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(\bar{a}, M_0, M_2)$  does not split over  $N$ . Hence  $\text{Av}_{<\kappa_1}(\mathbf{I}, M_0) = \text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(\bar{a}, M_0, M_2)$  does not split over  $N$ . Again as  $M_0 \prec_{\mathbb{L}_{\infty, \lambda}[\mathfrak{K}]} M_1$  we can deduce that  $\text{Av}_{<\kappa_1}(\mathbf{I}, M_1)$  does not split over  $N$  but by the choice of  $\mathbf{I}$  it is equal to  $\text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$ , so we are done.

(c)  $\Rightarrow$  (a)

By Claim 1.24 there is  $B \subseteq M_0$  of cardinality  $\leq \kappa_2$  such that  $\text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(\bar{a}, M_0, M_2)$  does not split over  $B$ .

As we can increase  $B$  as long as we preserve “of cardinality  $\leq \kappa_2$ ”, without loss of generality  $B = |N|$  where  $N \leq_{\mathfrak{K}} M_0$ . So the antecedent of clause (c) holds, but we are assuming clause (c) so the conclusion of clause (c) holds, that is  $\text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$  does not split over  $N$ .

Also by 1.26(1) there is  $\mathbf{I}_1 \subseteq {}^\alpha(M_0)$  of cardinality  $\kappa_2^+$  which is  $(\kappa_1^+, \kappa_1)$ -convergent and  $\text{Av}_{<\kappa_1}(\mathbf{I}_1, M_0) = \text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(\bar{a}, M_0, M_1)$ . Clearly  $\kappa_1 \geq \theta_1$  hence  $\kappa_2 = (\kappa_2)^{\theta_1}$ . Now as  $K_\lambda^*$  is categorical clearly  $M_0 \cong \text{EM}_{\tau(\mathfrak{K})}(\lambda, \Phi)$  hence applying 1.25(4) we can find  $\mathbf{I}_2 \subseteq \mathbf{I}_1$  of cardinality  $\kappa_2^+$  which is  $(\theta_1^+, \theta_1)$ -convergent. As above  $M_0 \prec_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]} M_1$  so we deduce that  $\mathbf{I}_2$  is  $(\theta_1^+, \theta_1)$ -convergent and  $(\kappa_1^+, \kappa_1)$ -convergent also in  $M_1$ .

As above we have  $M_0 \prec_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]} M_1$  by 1.19(1) hence  $\text{Av}_{<\kappa_1}(\mathbf{I}_2, M_1)$  is well defined and does not split over  $N$  hence is equal to  $\text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$ . This implies that  $\text{Av}_{<\theta_1}(\mathbf{I}_2, M_1) = \text{tp}_{\mathbb{L}_{\infty, \theta_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$ .

Now choose  $\mathbf{I}_3 \subseteq \mathbf{I}_2 \subseteq M_0$  of cardinality  $\theta_2$  and  $N_3 \leq_{\mathfrak{K}} M_0$  of cardinality  $\theta_2$  such that  $\mathbf{I}_3 \subseteq {}^\alpha(N_3)$ . Now by 1.23(2) we know that  $\text{tp}_{\mathbb{L}_{\infty, \theta_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$  does not split over  $\mathbf{I}_3$  hence it does not split over  $N_3$ , so  $N_3$  witnesses clause (a).  $\square_{4.6}$

**4.8 Definition.** We define a pre-frame  $\mathfrak{s}_\lambda = (\mathfrak{K}_{\mathfrak{s}_\lambda}, \bigcup_{\mathfrak{s}_\lambda}, \mathcal{S}_{\mathfrak{s}_\lambda}^{\text{bs}})$  as follows:

- (a)  $\mathfrak{K}_{\mathfrak{s}_\lambda} = \mathfrak{K}_\lambda^*$
- (b)  $\mathcal{S}_{\mathfrak{s}_\lambda}^{\text{bs}}$  is defined by  $\mathcal{S}_{\mathfrak{s}_\lambda}^{\text{bs}}(M) := \{\text{tp}_{\mathfrak{K}_\lambda^*}(a, M, N) : M \leq_{\mathfrak{K}_\lambda^*} N, a \in N \setminus M\}$ ,
- (c)  $\bigcup_{\mathfrak{s}_\lambda} = \{(M_0, M_1, a, M_3) : M_0 \leq_{\mathfrak{K}_\lambda^*} M_1 \leq_{\mathfrak{K}_\lambda^*} M_2 \text{ and } \text{tp}_{\mathfrak{K}_\lambda^*}(a, M_1, M_3) \text{ does not fork over } M_0\}$ , see Definition 4.5.

*4.9 Remark.* 1) Recall  $\leq_{\mathfrak{s}_\lambda} = \leq_{\mathfrak{K}} \upharpoonright K_{\mathfrak{s}_\lambda} = \leq_{\mathfrak{K}_\lambda^*}$ .

2) Concerning the proof of 4.10 below we mention a variant which the reader may ignore. This variant, from weaker assumptions gets weaker conclusions. In detail, define the weak versions  $(f)^-$  of  $(f)$  of 4.1(2); see Definition 1.34 and Claim 1.37(1)



$(f)^-$  if  $\langle M_\alpha : \alpha \leq \delta \rangle$  is  $\leq_{\mathfrak{K}}$ -increasing continuous and  $\alpha < \delta \Rightarrow M_{2\alpha+1} <_{\mathfrak{K}_\lambda^*}^* M_{2\alpha+2}$  (e.g.  $M_{2\alpha+2}$  is  $\leq_{\mathfrak{K}_\lambda^*}$ -universal over  $M_{2\alpha+1}$ ) hence both are from  $K_\lambda^*$  then  $M_\delta \in K_\lambda^*$ .

Assuming only 4.1(1) +  $(f)^-$  we do not know whether  $\mathfrak{K}_\lambda^*$  is a  $\lambda$ -a.e.c. but still  $(K_\lambda^*, \leq_{\mathfrak{K}} \upharpoonright K_\lambda^*, <_{\mathfrak{K}_\lambda^*}^*)$ , see Definition 1.34, is a so called semi  $\lambda$ -a.e.c., see Chapter N.

If clause  $(f)$  from 4.1(2) holds (i.e.,  $K_{\mathfrak{s}_\lambda}$  is closed under unions), we can omit “ $<_{\mathfrak{s}_\lambda}^*$ ”.

- 3) It will be less good but not a disaster if we have assumed below  $\lambda = \sup(\text{Cat}'_{\mathfrak{K}} \cap \lambda)$ .
- 4) It will be better to have  $\mathfrak{K}_{\mathfrak{s}_\lambda} = K_\lambda$ ; of courses, this follows from categoricity so by §3 is not unreasonable for conjecture 0.1.
- 5) But we can ask only for  $M \in K_{\mathfrak{s}_\lambda}$  to be universal in  $\mathfrak{K}_\lambda$ ,
- 6) We can ask that for every  $\mu > \lambda$  large enough, for every  $M \in K_\mu$  for a club of  $N \in K_\lambda$  satisfying  $N \leq_{\mathfrak{K}} M$  we have  $N \in K_{\mathfrak{s}_\lambda}$ .

**4.10 Theorem\*.** (Assume 4.1(2), (g) hence  $(f)$ ).  
 $\mathfrak{s}_\lambda$  is a good  $\lambda$ -frame categorical in  $\lambda$  and is full.

*Proof.* We check the clauses in the definition III.  600-1.1 .



Clause (A):

By observation 4.2(3), [in the weak version using (f)<sup>-</sup> from 4.9(1)].

Clause (B):

Categoricity holds by 1.16 (or 4.2(1)) and this implies “there is a superlimit model”, the non-maximality by  $\leq_{\mathfrak{K}_\lambda^*}$  holds by the choice of  $\Phi$ .

Clause (C):

Observation 4.2(2) guarantee amalgamation, categoricity (of  $\mathfrak{K}_\lambda^*$  by 4.2(1)) implies the JEP and “no-maximal model” holds by clause (B).

Clause (D)(a), (b):

Obvious by the definition.

(D) (c) (density).

Assume  $M <_{\mathfrak{K}_\lambda^*} N$ , then there are  $a \in N \setminus M$  and for any such  $a$  the type  $\mathbf{tp}_{\mathfrak{K}_\lambda^*}(a, M, N)$  belongs to  $\mathcal{S}_{\mathfrak{s}_\lambda}^{\text{bs}}(M)$ . In fact

⊗  $\mathfrak{s}_\lambda$  is type-full

(D) (d) (bs-stability).

The demand means  $M \in K_\lambda^* \Rightarrow |\mathcal{S}_{\mathfrak{K}_\lambda^*}^1(M)| \leq \lambda$ .

This holds by 1.32(2) (and amalgamation).

(E)(a), (b). By the definition.

(E)(c) (local character)

This says that if  $\langle M_i : i \leq \delta+1 \rangle$  is  $\leq_{\mathfrak{s}_\lambda}$ -increasing continuous and  $p = \mathbf{tp}_{\mathfrak{s}_\lambda}(a, M_\delta, M_{\delta+1}) \in \mathcal{S}_{\mathfrak{s}_\lambda}^{\text{bs}}(M_\delta)$  then for some  $i < \delta$  the type  $p$  does not fork over  $M_i$  (for  $\mathfrak{s}_\lambda$ ).

From now on (in the proof of 4.10) we use 4.6 freely and let (noting  $\text{cf}(\delta) < \lambda$  as  $\lambda$  is singular)

$$\odot \quad \kappa_0 = \text{LS}(\mathfrak{K}) + \text{cf}(\delta), \kappa_1 = \beth_{1,1}(\beth_2(\kappa_0))^+, \kappa_2 = \beth_2(\kappa_1).$$

Now by 4.6 there is a  $(\kappa_1^+, \kappa_1)$ -convergent  $\mathbf{I} \subseteq M_\delta$  with  $\text{Av}_{<\kappa_1}(\mathbf{I}, M_\delta) = \text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(a, M_\delta, M_{\delta+1})$  such that  $\mathbf{I}$  is of cardinality  $> \kappa_2$ . For some  $i(*) < \delta$ ,  $|\mathbf{I} \cap M_{i(*)}| > \kappa_2$ , so without loss of generality  $\mathbf{I} \subseteq M_{i(*)}$ , so by 4.6 we are done.

(E)(d) Transitivity of non-forking



We are given  $M_0 \leq_{\mathfrak{s}_\lambda} M_1 \leq_{\mathfrak{s}_\lambda} M_2 \leq_{\mathfrak{K}_\mathfrak{s}} M_3$  and  $a \in M_3$  such that  $\mathbf{tp}_{\mathfrak{s}_\lambda}(a, M_{\ell+1}, M_3)$  does not fork over  $M_\ell$  for  $\ell = 0, 1$ . So for  $\ell = 0, 1$  there is  $\mathbf{I}_\ell \subseteq M_\ell$  which is  $(\kappa_1^+, \kappa_1)$ -convergent in  $M_{\ell+1}$  of cardinality  $\kappa_2^+$  such that  $\text{Av}_{<\kappa_1}(\mathbf{I}_\ell, M_{\ell+1}) = \text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(a, M_{\ell+1}, M_3)$ . As  $\text{Av}_{<\kappa_1}(\mathbf{I}_0, M_1) = \text{Av}_{<\kappa_1}(\mathbf{I}_1, M_1)$  (being both realized by  $a$ ) because  $M_1 \prec_{\mathbb{L}_{\infty, \lambda}[\mathfrak{K}]} M_2$  by 4.4(4) clearly we have  $\text{Av}_{<\kappa_1}(\mathbf{I}_0, M_2) =$

$\text{Av}_{<\kappa_1}(\mathbf{I}_1, M_2) = \text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(a, M_2, M_3)$  all well defined. So  $\mathbf{I}_0$  witness by 4.6 that  $\text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(a, M_2, M_3)$  does not fork over  $M_0$ , which means that  $\mathbf{tp}_{\mathfrak{K}_\lambda^*}(a, M_2, M_3)$  does not fork over  $M_0$  as required.

(E)(e) Uniqueness.

Recalling 4.4(1), the proof is similar to (E)(d); the two witnesses are now in  $M_0$ .

(E)(f) Symmetry

Toward contradiction, recalling III.  600-1.16E  assume  $M_0 \leq_{\mathfrak{K}_\lambda^*} M_1 \leq_{\mathfrak{K}_\lambda^*} M_2 \leq_{\mathfrak{K}_\lambda^*} M_3$  and  $a_\ell \in M_{\ell+1} \setminus M_\ell$  for  $\ell = 0, 1, 2$  are such that  $p_\ell = \mathbf{tp}_{\mathfrak{K}_\lambda^*}(a_\ell, M_\ell, M_{\ell+1})$  does not fork over  $M_0$  for  $\ell = 0, 1, 2$  and  $\mathbf{tp}_{\mathfrak{K}_\lambda^*}(a_0, M_0, M_1) = \mathbf{tp}_{\mathfrak{K}_\lambda^*}(a_2, M_0, M_3)$  but  $\mathbf{tp}_{\mathfrak{K}_\lambda^*}(\langle a_0, a_1 \rangle, M_0, M_3) \neq \mathbf{tp}_{\mathfrak{K}_\lambda^*}(\langle a_2, a_1 \rangle, M_0, M_3)$ .

By 4.6 we can deal with  $p_\ell = \text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(a_\ell, M_\ell, M_{\ell+1})$  for  $\ell = 0, 1, 2$ . For each  $\ell \leq 2$ , we can find convergent  $\mathbf{I}_\ell = \{a_\alpha^\ell : \alpha < \kappa_2^+\} \subseteq M_0$  which is  $(\kappa_1^+, \kappa_1)$ -convergent such that  $\text{Av}_{<\kappa_1}(\mathbf{I}_\ell, M_\ell) = p_\ell$ .



So as  $M_0 \prec_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]} M_k$  we deduce the set  $\mathbf{I}_\ell$  is  $(\kappa_1^+, \kappa_1)$ -convergent in  $M_k$  for  $\ell, k = 0, 1, 2$ , also  $\text{Av}_{<\kappa_1}(\mathbf{I}_0, M_0) = \text{Av}_{<\kappa_1}(\mathbf{I}_2, M_0)$  hence  $\text{Av}_{<\kappa_1}(\mathbf{I}_0, M_2) = \text{Av}_{<\kappa_1}(\mathbf{I}_2, M_2)$  so without loss of generality  $\mathbf{I}_0 = \mathbf{I}_2$ .

Now use the non-order property to get symmetry.

(E)(g) Existence

So assume  $M \leq_{\mathfrak{s}_\lambda} N$  and  $p \in \mathcal{S}_{\mathfrak{s}_\lambda}^{\text{bs}}(M)$ . So we can find a pair  $(M', a)$  such that  $M \leq_{\mathfrak{s}_\lambda} M', a \in M_1$  and  $p = \mathbf{tp}_{\mathfrak{s}_\lambda}(a, M, M')$ . By 1.26(1) there is a  $(\kappa_1^+, \kappa_1)$ -convergent  $\mathbf{I} \subseteq M$  of cardinality  $\kappa_2^+$  such that  $\text{Av}_{<\kappa_1}(M, \mathbf{I}) = \text{tp}_{\mathbb{L}_{\infty, \kappa_1}[\mathfrak{K}]}(a, M, M')$ . By 1.26(3) + 4.6 there is a pair  $(N', a')$  such that  $N \leq_{\mathfrak{s}_\lambda} N', a' \in N'$  and  $\text{tp}_{\mathbb{L}_{\infty, \kappa_1}}(a', N, N') = \text{Av}_{<\kappa_1}(\mathbf{I}, N)$ . So by 4.6 the type  $\mathbf{tp}_{\mathfrak{s}_\lambda}(a', N, N')$  easily  $\in \mathcal{S}_{\mathfrak{s}_\lambda}^{\text{bs}}(N)$ , does not fork over  $N$  and extend  $p$ , as required.

(E)(h) Continuity

Follow by III.  600-1.16A  Alternatively assume  $\langle M_i : i \leq \delta + 1 \rangle$  is  $\leq_{\mathfrak{s}_\lambda}$ -increasing continuous, and  $a \in M_{\delta+1} \setminus M_\delta$  and  $\mathbf{tp}_{\mathfrak{s}_\lambda}(a, M_i, M_{\delta+1})$  does not fork over  $M_0$  for  $i < \delta$ . So there is a convergent  $\mathbf{I}_i \subseteq M_0$  such that  $i < \delta \Rightarrow \text{tp}_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]}(a, M_i, M_{\delta+1}) = \text{Av}_\kappa(\mathbf{I}_i, M_i)$ .

As above, without loss of generality  $\mathbf{I}_i = \mathbf{I}_0$ . We can find a convergent  $\mathbf{I} \subseteq M_\delta$  of cardinality  $> \text{cf}(\delta) + \kappa$  (recall  $\text{cf}(\delta) < \lambda$ !) such that  $\text{tp}_{\mathbb{L}_{\infty, \kappa}[\mathfrak{K}]}(a, M_0, M_{\delta+1}) = \text{Av}_\kappa(\mathbf{I}, M_\delta)$ . So for some  $i(*) < \delta$ ,  $|\mathbf{I} \cap M_{i(*)}| > \kappa$  so without loss of generality (by equivalence)  $\mathbf{I} \subseteq M_{i(*)}$ . We finish as in (E)(f).

Axiom (E)(i):

Follows by III.  600-1.15 .

$\square_{4.10}$

4.11 Exercise: Replace above  $\text{Av}_{<\kappa_1}(\mathbf{I}, M)$  by  $\cup \{ \text{Av}_{\beth_\zeta(\kappa_0)}(\mathbf{I}, M) : \zeta < (2^{\kappa_0})^+ \}$ .

## §5 HOMOGENEOUS ENOUGH LINEAR ORDERS

**5.1 Claim.** Assume  $\mu^+ = \theta_1 = \text{cf}(\theta_1) < \theta_2 = \text{cf}(\theta_2) < \lambda$ .

1) Then there is a linear order  $I$  of cardinality  $\lambda$  such that: the following equivalence relation  $\mathcal{E} = \mathcal{E}_{I,\mu}^{\text{aut}}$  on  ${}^\mu I$  has  $\leq 2^\mu$  equivalence classes, where

$\eta_1 \mathcal{E} \eta_2$  iff there is an automorphism of  $I$  mapping  $\eta_1$  to  $\eta_2$ .

2) Moreover if  $I' \subseteq I$  has cardinality  $< \theta_2$  and  $n < \omega$  then the following equivalence relation  $\mathcal{E}$  on  ${}^n I$  has  $\leq \mu + |I'|$  equivalence classes:

$\bar{s} \mathcal{E} \bar{t}$  iff there is an automorphism  $h$  of  $I$  over  $I'$  mapping  $\bar{s}$  to  $\bar{t}$ .

3) Moreover, there is  $\Psi$  proper for  $K_{\tau_2^*}^{\text{lin}}$  (i.e.  $\Psi \in \Upsilon_{\aleph_0}^{\text{lin}}[2]$ , see Definition 0.11(5), 0.14(9)) with  $\tau(\Psi)$  countable such that  $I = \text{EM}_{\{<\}}(I_{\theta_2, \lambda \times \theta_2}^{\text{lin}}, \Phi)$  where  $I_{\theta_2, \zeta}^{\text{lin}} = (\zeta, <, P_0, P_1), P_\ell = \{\alpha < \zeta : (\text{cf}(\alpha) < \theta_2) \equiv (\ell = 0)\}$ .

4) If  $I_0^* \subseteq I$  has cardinality  $< \theta_2$  then for some  $I_1^* \subseteq I$  of cardinality  $\leq \mu^+ + |I_0^*|$  for every  $J \subseteq I$  of cardinality  $\leq \mu$  there is an automorphism of  $I$  over  $I_0^*$  mapping  $J$  into  $I_1^*$ .

5) If  $I_1^*, I_2^* \subseteq I_{\mu, \lambda \times \mu^+}^{\text{lin}}$  has cardinality  $\leq \mu$  and  $h$  is an isomorphism from  $I_1^*$  onto  $I_2^*$  then there is an automorphism  $\hat{h}$  of the linear order  $I = \text{EM}_{\{<\}}(I_{\theta, \lambda}^{\text{lin}}, \Psi)$  extending the natural isomorphism  $\check{h}$  from  $\text{EM}_{\{<\}}(I_1^*, \Psi)$  onto  $\text{EM}_{\{<\}}(I_2^*, \Psi)$ .

*Remark.* 1) Of course, if  $\lambda = \lambda^{<\theta_2}$  and  $I$  is a dense linear order of cardinality  $\lambda$  which is  $\theta$ -strongly saturated (hence  $\theta$ -homogeneous) then the demand in 5.1(1) is satisfied (and in part (2) of 5.1 the number (of  $\mathcal{E}$  equivalence classes) is  $\leq 2^\chi$  for every  $\chi \in [\aleph_0, \theta_2)$ ). Also if  $\lambda = \sum_{i < \delta} \lambda_i, \delta < \theta_2$  and  $i < \delta \Rightarrow \lambda_i^{<\theta_2} = \lambda$  we have such order.

2) Laver [Lv71, §2] deals with related linear orders but for his aims  $I_1, I_2$  are equivalent if each is embeddable into the other; see more in [Sh:e, AP, §2]. For a cardinal  $\partial$  and linear order  $I$  let  $\Theta_{I, \partial} = \{\text{cf}(J) : \text{for some } <_I\text{-decreasing sequence } \langle t_i : i < \partial \rangle \text{ we have } J = I \upharpoonright \{t \in I : t <_I t_i \text{ for every } i < \partial\}\}$ . So if  $\partial \leq \mu$  then  $({}^\mu I)/E_{I, \mu}^{\text{aut}}$  has  $\geq |\Theta_{I, \partial}|$ . So we have to be careful to make  $\Theta_{I, \partial}$  small. We choose a very concrete construction which leads quickly to defining  $I$  and the checking is straight so we thought it would be easy but a posteriori the checking is lengthy; [Sh:e, AP, §2] is an anti-thetical approach.

3) We can replace  $\theta_1 = \mu^+$  by  $\theta_1 = \text{cf}(\theta_1) > \aleph_0$  and “of cardinality  $\leq \mu$ ” by “of cardinality  $< \theta_1$ ”.

4) In 2.8(1), 2.11(2) we use parts (1), (1)+(4) respectively. Also we use 5.1 in the proof of 7.8.

5) The case  $2^\mu \geq \lambda$  in 5.1(1) says nothing, in fact if  $2^\mu \geq \lambda$  then  $2^\mu = \lambda^\mu =$

$(^\mu M)/\mathcal{E}_{I,\mu}^{\text{aut}}$  for any model  $M$  of cardinality  $\leq 2^\mu$  but  $\geq 2$ , for any vocabulary  $\tau_M$ .  
 6) Claim 5.1(1),(2) holds also if we replace  $\mu$  by  $\chi \in [\mu, \theta_2)$ .

*Proof.* 1) Fix an ordinal  $\zeta, \lambda \leq \zeta < \lambda^+$  such that  $\text{cf}(\zeta) = \theta_2$ , e.g.,  $\zeta = \lambda \times \theta_2$  (almost always  $\text{cf}(\zeta) \geq \theta_2$  suffice).

Let  $I_1$  be the following linear order, its set of elements is  $\{(\ell, \alpha) : \ell \in \{-2, -1, 1, 2\}, \alpha < \zeta + \omega\}$  ordered by  $(\ell_1, \alpha_1) <_{I_1} (\ell_2, \alpha_2)$  iff  $\ell_1 < \ell_2$  or  $\ell_1 = \ell_2 \in \{-1, 2\} \wedge \alpha_1 < \alpha_2$  or  $\ell_1 = \ell_2 \in \{-2, 1\} \wedge \alpha_1 > \alpha_2$ .

For  $t \in I_1$  let  $t = (\ell^t, \alpha^t)$ .

Let  $I_2^*$  be the set  $\{\eta : \eta \text{ is a finite sequence of members of } I_1\}$  ordered by  $\eta_1 <_{I_2} \eta_2$  iff  $(\exists n)(n < \ell g(\eta_1) \wedge n < \ell g(\eta_2) \wedge \eta_1 \upharpoonright n = \eta_2 \upharpoonright n \ \& \ \eta_1(n) <_{I_1} \eta_2(n))$  or  $\eta_1 \triangleleft \eta_2 \wedge \ell^{\eta_2(\ell g(\eta_1))} \in \{1, 2\}$  or  $\eta_2 \triangleleft \eta_1 \wedge \ell^{\eta_1(\ell g(\eta_2))} \in \{-2, -1\}$ .

Let  $I_2$  be  $I_2^*$  restricted to the set of  $\eta \in I_2^*$  satisfying  $\circledast$  where

$\circledast$  for no  $n < \omega$  do we have:

- (a)  $\ell g(\eta) > n + 1$
- (b)  $\alpha^{\eta(n)}$  is a limit ordinal of cofinality  $\geq \theta_1$
- (c)  $\alpha^{\eta(n+1)} \geq \zeta$
- (d)  $\ell^{\eta(n)} \in \{-1, 2\}, \ell^{\eta(n+1)} = -2$  or  $\ell^{\eta(n)} \in \{-2, 1\}, \ell^{\eta(n+1)} = 2$ .

Let  $M_0$  be the following ordered field:

- (\*)<sub>1</sub> (a)  $M_0$  as a field, is  $\mathbb{Q}(a_t : t \in I_2)$ , the field of rational functions with  $\{a_t : t \in I_2\}$  algebraically independent
- (b) the order of  $M_0$  is determined by
  - ( $\alpha$ ) if  $t \in I_2, n < \omega$  then  $M_0 \models n < a_t$
  - ( $\beta$ ) if  $s <_{I_2} t$  and  $n < \omega$  then  $M_0 \models "(a_s)^n < a_t"$ .
- (c) let  $M$  be the real<sup>3</sup> (algebraic) closure of  $M_0$  (i.e. the elements algebraic over  $M_0$  in the closure by adding elements realizing any Dedekind cut of  $M_0$ ).

Now we shall prove that  $I$ , which is  $M$  as a linear order, is as requested.

$\boxtimes_1$  each of  $I_1, I_2^*$  and  $I_2$  is anti-isomorphic to itself.

---

<sup>3</sup>in fact, we could just use  $M_0$

[Why? Let  $g : I_1 \rightarrow I_1$  be  $g(t) = (-\ell^t, \alpha^t)$ , clearly it is an anti-isomorphism of  $I_1$ . Let  $\hat{g} : I_2^* \rightarrow I_2^*$  be defined by  $\hat{g}(\eta) = \langle g(\eta(m)) : m < \ell g(\eta) \rangle$ , it is an anti-isomorphism of  $I_2^*$ . Lastly  $\hat{g}$  maps  $I_2$  onto itself, in particular by the character of clause (d) of  $\otimes$ , i.e. the two cases are interchanged by  $\hat{g}$ ]

- $\boxtimes_2$  (a)  $I_1, I_2^*, I_2$  have cofinality  $\aleph_0$ .  
 (b) if  $t \in I_2$  then  $I_{2, < t} := I_2 \upharpoonright \{s : s <_{I_2} t\}$  has cofinality  $\aleph_0$ .

[Why? For clause (a),  $\{(2, \lambda + n) : n < \omega\}$  is a cofinal subset of  $I_1$  of order type  $\omega$  and  $\{< t > : t \in I_1\}$  is a cofinal subset of  $I_2^*$  and of  $I_2$  of order type being the same as  $I_1$ . For clause (b) for  $\eta \in I_2$  the set  $\{\eta^\wedge \langle (-1, \lambda + n) \rangle : n < \omega\}$  is a cofinal subset of  $I_{2, < \eta}$  of order type  $\omega$  by  $\square$  below.]

Now

- $\square$  if  $\eta$  satisfies  $\otimes$  and  $\ell \in \{1, -1\}$  then also  $\eta^\wedge \langle (\ell, \alpha) \rangle$  satisfies  $\otimes$  for any  $\alpha < \lambda + \omega$ .

[Why? By clause (d) of  $\otimes$  as the only value of  $n$  there which is not obvious is  $n = \ell g(\eta) - 1$ , but to be problematic we should have  $\ell^{(\eta^\wedge \langle (\ell, \alpha) \rangle)(n+1)} \in \{-2, 2\}$  whereas  $\ell = -1$ .]

- $\boxtimes_3$  if  $\partial = \text{cf}(\partial)$  so  $\partial$  is 0, 1 or an infinite regular cardinal and  $\bar{\eta} = \langle \eta_i : i < \partial \rangle$  is a  $<_{I_2}$ -decreasing sequence and we let  $J_{\bar{\eta}} = \{s \in I_2 : s <_{I_2} \eta_i \text{ for every } i < \partial\}$  then (clearly exactly one of the following clauses applies)
- (a) if  $J_{\bar{\eta}} = \emptyset$  then  $\partial = \aleph_0$
  - (b) if  $\text{cf}(J_{\bar{\eta}}) = 1$  then  $\partial = \aleph_0$
  - (c) if  $\text{cf}(J_{\bar{\eta}}) = \aleph_0$  then  $\partial < \theta_1$
  - (d) if  $\aleph_1 \leq \text{cf}(J_{\bar{\eta}}) < \theta_1$  then  $\partial = \aleph_0$  and for some  $\ell \in \{-1, 2\}, \nu \in I_2$  and ordinal  $\delta < \zeta$  of cofinality  $\text{cf}(J_{\bar{\eta}})$  the set  $\langle \nu^\wedge \langle (\ell, \alpha) \rangle : \alpha < \delta \rangle$  is an unbounded subset of  $J_{\bar{\eta}}$
  - (e) if  $\theta_1 \leq \text{cf}(J_{\bar{\eta}})$  then  $\partial \geq \theta_1$  and moreover  $\partial = \theta_2 \vee \text{cf}(J_{\bar{\eta}}) = \theta_2$ .

[Why does  $\boxtimes_3$  hold? The proof is split into cases and finishing a case we can then assume it does not occur.

Clearly we can replace  $\bar{\eta}$  by  $\langle \eta_i : i \in u \rangle$  for any unbounded subset  $u$  of  $\partial$  and by  $\langle \nu_i : i \in u \rangle$  if  $\eta_{\zeta_{2i+1}} \leq_{I_2} \nu_i \leq_{I_2} \eta_{\zeta_{2i}}$  and  $\langle \zeta_i : i < \partial \rangle$  an increasing sequence of ordinals  $< \partial$ . We shall use this freely.

Case 0:  $\partial = 0$  or  $\partial = 1$ .

By  $\boxtimes_2$  clearly clause (c) of  $\boxtimes_3$  holds.

Case 1:  $\partial = \aleph_0$  and there is  $\nu \in {}^\omega(I_1)$  such that  $(\forall n < \omega)(\exists i < \partial)(\eta_i \upharpoonright n \triangleleft \nu)$ .

Let  $n_i = \ell g(\eta_i \cap \nu)$ , it is impossible that  $\{i : n_i = k\}$  is infinite for some  $k$ , so without loss of generality  $\langle n_i : i < \omega \rangle$  is an increasing sequence and  $n_0 > 0$ .

For every  $i < \omega$  we have  $\nu \upharpoonright (n_i + 1) \leq \eta_{i+1}$  and  $\eta_{i+1} <_{I_2} \eta_i$ , so by the definition of  $<_{I_2}$  also  $\nu \upharpoonright (n_i + 1) <_{I_2} \eta_i$ , and we choose  $\beta_{n_i} < \zeta + \omega$  so that  $(-2, \beta_{n_i}) <_{I_1} \nu(n_i)$  hence letting  $\rho_i = \nu \upharpoonright n_i \wedge \langle (-2, \beta_{n_i}) \rangle$  we have  $\rho_i \in I_2$ . This can be done, e.g. because we can choose  $\beta_{n_i}$  such that  $\beta_{n_i} = \alpha^{\nu(n_i)} + 1$  if  $\ell^{\nu(n_i)} = -2$  and  $\beta_{n_i} = 0$  otherwise.

For every  $i, j < \omega$  we have  $\rho_i <_{I_2} \rho_{i+1} <_{I_2} \eta_{i+1} <_{I_2} \eta_i$ , so if  $i \leq j$  then  $\rho_i <_{I_2} \rho_j <_{I_2} \eta_j$ , and if  $i > j$  then  $\rho_i <_{I_2} \eta_i <_{I_2} \eta_j$ , so  $\rho_i \in J_{\bar{\eta}}$ .

Now  $\langle \rho_i : i < \omega \rangle$  is  $<_{I_2}$ -increasing also it is cofinal in  $J_{\bar{\eta}}$ , for if  $\rho \in J_{\bar{\eta}}$  let  $n = \ell g(\rho \cap \nu)$ , so for  $i < \omega$  such that  $n_i \leq n < n_{i+1}$  we have  $\rho <_{I_2} \eta_{i+1}$  so  $\rho(n) <_{I_1} \eta_{i+1}(n) = \rho_{i+1}(n)$  and as  $\rho \upharpoonright n = \nu \upharpoonright n = \rho_{i+1} \upharpoonright n$  we have  $\rho <_{I_2} \rho_{i+1}$ .

As  $\langle \rho_i : i < \omega \rangle$  is of order type  $\omega$  clearly  $\text{cf}(J_{\bar{\eta}}) = \aleph_0 = \partial$  hence clause (c) of  $\boxtimes_3$  applies, and we are done.

So from now on assume that case 1 fails.

As  $\ell g(\eta_i) < \omega$  and as not Case 1 without loss of generality for some  $n$ , we have  $i < \partial \Rightarrow \ell g(\eta_i) = n$ . Similarly without loss of generality for some  $m$  and  $\nu \in I_2$  we have  $i < \partial \Rightarrow \eta_i \upharpoonright m = \nu$  and  $\langle \eta_i(m) : i < \partial \rangle$  with no repetitions so  $m < n$ . Without loss of generality  $i < \partial \Rightarrow \ell^{\eta_i(m)} = \ell^*$  and so  $\langle \alpha^{\eta_i(m)} : i < \partial \rangle$  is with no repetitions; and without loss of generality is monotonic hence, as  $\partial \geq \aleph_0$  is an increasing sequence of ordinals. As  $\bar{\eta}$  is  $<_{I_2}$ -decreasing necessarily  $\ell^* \in \{-2, 1\}$  and let  $\delta = \cup \{\alpha^{\eta_i(m)} : i < \partial\}$ , so clearly  $\text{cf}(\delta) = \partial$  and  $\delta$  is a limit ordinal  $\leq \zeta + \omega$ . Now those  $\ell^*, \delta$  will be used till the end of the proof of  $\boxtimes_3$ . So for the rest of the proof we are assuming

- ⊙ (a)  $i < \partial \Rightarrow \eta_i \upharpoonright m = \nu$
- (b)  $\langle \eta_i(m) : i < \partial \rangle$  is (strictly) increasing with limit  $\delta$
- (c)  $\ell^{\eta_i(m)} = \ell^* \in \{-2, 1\}$
- (d)  $\text{cf}(\delta) = \partial, \delta \leq \zeta + \omega$ .

Also note by  $\otimes$  that  $\nu \wedge \langle (\ell^*, \delta) \rangle \notin I_2 \Rightarrow \delta \in \{\zeta + \omega, \zeta\}$  and if  $\delta = \zeta \wedge \nu \wedge \langle (\ell^*, \delta) \rangle \notin I_2$  then  $\ell g(\nu) > 0$  and the ordinal  $\alpha^{\nu(\ell g(\nu)-1)}$  is limit of cofinality  $\geq \theta_1$  (and more).

Case 2:  $J_{\bar{\eta}} = \emptyset$ .

Clearly  $m = 0 \wedge \ell^* = -2 \wedge \delta = \zeta + \omega$  hence  $\partial = \aleph_0$  so clause (a) of  $\boxtimes_3$  holds.

Case 3:  $\ell^* = 1$  and  $\nu \wedge \langle (\ell^*, \delta) \rangle \notin I_2$ .

As  $\ell^* = 1$  clearly we cannot have  $\delta = \zeta$  by clause (d) of  $\otimes$  so  $\delta = \zeta + \omega$  and recalling  $\partial = \text{cf}(\delta)$  we have  $\partial = \aleph_0$ . Now clearly  $J_{\bar{\eta}}$  has a last element,  $\nu$ , so case (b) of  $\boxtimes_3$  applies.

Case 4:  $\ell^* = -2$ ,  $\partial = \aleph_0$  and  $\nu \wedge \langle (\ell^*, \delta) \rangle \notin I_2$ .

Again  $\delta = \zeta + \omega$  as  $\aleph_0 = \partial = \text{cf}(\delta)$  and  $\text{cf}(\zeta) = \theta_2 > \mu \geq \aleph_0$  making  $\delta = \zeta$  impossible; now  $\ell g(\nu) > 0$  (as we have discarded the case  $J_{\bar{\eta}} = \emptyset$ , i.e. Case 2); and let  $k = \ell g(\nu) - 1$ . Now we prove case 4 by splitting to several subcases.

Subcase 4A:  $\ell^{\nu(k)} \in \{-2, 1\}$ .

Let  $\nu_1 = (\nu \upharpoonright k) \wedge \langle (\ell^{\nu(k)}, \alpha^{\nu(k)} + 1) \rangle$ , note that  $\nu_1 \in I_2$  as  $\nu \in I_2 \wedge (\alpha^{\nu(k)} < \zeta \equiv \alpha^{\nu(k)} + 1 < \zeta)$  and (as  $\ell^{\nu(k)} \in \{-2, 1\}$ ) clearly  $\{\rho : \nu_1 \trianglelefteq \rho \in I_2\}$  is a cofinal subset of  $J_{\bar{\eta}}$  even an end segment. Now for  $n < \omega$  we have  $\nu_1 \wedge \langle (2, \zeta + n) \rangle \in I_2^*$  and it satisfies  $\circledast$ . (Why? As  $\nu_1 \in I_2$ , only  $n = k$  may be problematic, but  $\alpha^{\nu(k)} + 1 = \alpha^{\nu_1(k)}$  here stands for  $\alpha^{\eta(n)}$  there hence clause (b) of  $\circledast$  does not apply), so by the definition of  $I_2$ , clearly  $\{\nu_1 \wedge \langle (2, \zeta + n) \rangle : n < \omega\}$  is  $\subseteq I_2$  and is a cofinal subset of  $J_{\bar{\eta}}$  so  $\partial = \aleph_0 = \text{cf}(J_{\bar{\eta}})$  and clause (c) of  $\boxtimes_3$  holds.

Subcase 4B:  $\ell^{\nu(k)} \in \{-1, 2\}$  and  $\alpha^{\nu(k)}$  is a successor ordinal.

Let  $\nu_1 = (\nu \upharpoonright k) \wedge \langle (\ell^{\nu(k)}, \alpha^{\nu(k)} - 1) \rangle$ , of course  $\nu_1 \in I_2^*$  and as  $\nu \in I_2$  clearly  $\nu_1 \in I_2$  so the set  $\{\rho : \nu_1 \trianglelefteq \rho \in I_2\}$  is an end segment of  $J_{\bar{\eta}}$  and has cofinality  $\aleph_0$  because  $n < \omega \Rightarrow \nu_1 \wedge \langle (2, \zeta + n) \rangle \in I_2$ . (Why? It  $\in I_2^*$  and as  $\nu_1 \in I_2$  checking  $\circledast$  only  $n = k$  may be problematic, but  $(\ell^{\nu(k)}, 2)$  here stand for  $(\ell^{\eta(n)}, \ell^{\eta(n+1)})$  there but presently  $\ell^{\nu(k)} \in \{-1, 2\}$  contradicting clause (d) of  $\circledast$ ). So clause (c) of  $\boxtimes_3$ .

Subcase 4C:  $\ell^{\nu(k)} \in \{-1, 2\}$  and  $\alpha^{\nu(k)} = 0$ .

Then let  $\nu_1 = (\nu \upharpoonright k) \wedge \langle (\ell^{\nu(k)} - 1, 0) \rangle$ . Now  $\nu_1 \in I_2$  as  $\nu \upharpoonright k \in I_2$  and for  $n = k - 1$  clause (c) of  $\circledast$  fails and  $\nu_1 \wedge \langle (2, \zeta + n) \rangle \in I_2$  because of  $\nu_1 \in I_2$  and for  $n = k$  the failure of clause (b) of  $\circledast$  so continue as in Subcase 4B above.

Lastly,

Subcase 4D:  $\ell^{\nu(k)} \in \{-1, 2\}$  and  $\alpha^{\nu(k)}$  is a limit ordinal.

Then  $\{(\nu \upharpoonright k) \wedge \langle (\ell^{\nu(k)}, \alpha) \rangle : \alpha < \alpha^{\nu(k)}\}$  is  $\subseteq I_2$  and is an unbounded subset of  $J_{\bar{\eta}}$  hence  $\text{cf}(J_{\bar{\eta}}) = \text{cf}(\alpha^{\nu(k)})$ . If  $\text{cf}(\alpha^{\nu(k)}) = \aleph_0$ , then clause (c) in  $\boxtimes_3$  holds, and if  $\text{cf}(\alpha^{\nu(k)}) \in [\aleph_1, \theta_1)$  then necessarily  $\alpha^{\nu(k)} \neq \zeta$  so being a limit ordinal  $< \zeta + \omega$  clearly  $\alpha^{\nu(k)} < \zeta$  so clause (d) from  $\boxtimes_3$  holds. To finish this subcase note that  $\text{cf}(\alpha^{\nu(k)}) \geq \theta_1$  is impossible.

[Why “impossible”? Clearly for large enough  $i < \partial$  we have  $\eta_i(m) \geq \zeta$  (because  $\delta = \zeta + \omega$  as said in the beginning of the case) and recall  $\nu \triangleleft \eta_i \in I_2$ . We now show that clauses (a)-(d) of  $\circledast$  hold with  $\eta_i, k$  here standing for  $\eta, n$  there. For clause (a) recall  $\ell g(\eta_i) \geq \ell g(\nu) + 1$  and  $m = \ell g(\nu) = k + 1$ . Now  $\ell^{\eta_i(k+1)} = \ell^{\eta_i(m)} = \ell^* = -2$  as  $\ell^* = -2$  is part of the case,  $\ell^{\eta_i(k)} = \ell^{\nu(k)} \in \{-1, 2\}$  in this subcase, so clause (d) of  $\circledast$  holds. Also  $\alpha^{\eta_i(k+1)} = \alpha^{\eta_i(m)} \geq \zeta$  as said above so clause (c) of  $\circledast$  holds and  $\text{cf}(\alpha^{\eta_i(k)}) = \text{cf}(\alpha^{\nu(k)}) \geq \theta_1$  (as we are trying to prove “impossible”), so clause (b) of  $\circledast$  holds. Together we have proved (a)-(d) of  $\circledast$ . But  $\eta_i \in I_2$ , contradiction.]

Now subcases 4A,4B,4C,4D cover all the possibilities hence we are done with case 4.

Case 5:  $\ell^* = -2$ ,  $\partial > \aleph_0$  and  $\nu^\wedge \langle (\ell^*, \delta) \rangle \notin I_2$ .

Recalling  $\delta$  is the limit of the increasing sequence  $\langle \alpha^{\eta_i(m)} : i < \partial \rangle$  hence  $\text{cf}(\delta) = \partial > \aleph_0$  and  $\nu^\wedge \langle (-2, \delta) \rangle \notin I_2$ , necessarily  $\delta = \zeta$  so  $\partial = \theta_2$ . As  $\nu^\wedge \langle (-2, \delta) \rangle \notin I_2$  necessarily clauses (a) - (d) of  $\circledast$  hold for some  $n$  and as  $\nu \in I_2$ , clearly  $n = \ell g(\nu) - 1$  (see clause (a) of  $\circledast$ ) so we have  $\ell g(\nu) > 0$ , and letting  $k = \ell g(\nu) - 1$ , by clause (d) of  $\circledast$  the  $\ell^{\eta(n+1)}$  there stands for  $\ell^* = -2$  here so we have  $\ell^{\nu(k)} \in \{-1, 2\}$  and by clause (b) of  $\circledast$  we have  $\text{cf}(\alpha^{\nu(k)}) \geq \theta_1$ . Hence  $\{(\nu \upharpoonright k)^\wedge \langle (\ell^{\nu(k)}, \beta) \rangle : \beta < \alpha^{\nu(k)}\}$  is cofinal in  $J_{\bar{\eta}}$  and its cofinality is  $\text{cf}(\alpha^{\nu(k)})$  as  $(\nu \upharpoonright k)^\wedge \langle (\ell^{\nu(k)}, \beta) \rangle$  increase (by  $\leq_{I_2}$ ) with  $\beta$  as  $\ell^{\nu(k)} \in \{-1, 2\}$ . But  $\text{cf}(\alpha^{\nu(k)}) \geq \theta_1$  and  $\partial = \theta_2$  (see first sentence of the present case), so clause (e) of  $\boxtimes_3$  holds.

Case 6:  $\nu^\wedge \langle (\ell^*, \delta) \rangle \in I_2$ .

Subcase 6A:  $\nu^\wedge \langle (\ell^*, \delta), (2, \zeta) \rangle \in I_2$ .

Note that for  $m = \ell g(\nu)$  and the pair  $(\nu^\wedge \langle (\ell^*, \delta), (2, \zeta) \rangle, m)$  standing for  $(\eta, n)$  in  $\circledast$ , clauses (a),(c),(d) of  $\circledast$  hold (recall  $\ell^* \in \{-2, 1\}$ , see the discussion after case 1) so necessarily clause (b) of  $\circledast$  fails hence  $\text{cf}(\delta) < \theta_1$  but  $\partial = \text{cf}(\delta)$  so  $\partial < \theta_1$ . Now as  $\nu^\wedge \langle (\ell^*, \delta), (2, \zeta) \rangle \in I_2$  clearly if  $\ell < \omega$ , then  $\nu^\wedge \langle (\ell^*, \delta), (2, \zeta + \ell) \rangle$  belongs to  $I_2$  hence  $\{\nu^\wedge \langle (\ell^*, \delta), (2, \zeta + \ell) \rangle : \ell < \omega\}$  is a cofinal subset of  $J_{\bar{\eta}}$  by the choice of  $I_2$  hence  $\text{cf}(J_{\bar{\eta}}) = \aleph_0$  so clause (c) of  $\boxtimes_3$  applies.

Subcase 6B:  $\nu^\wedge \langle (\ell^*, \delta), (2, \zeta) \rangle \notin I_2$ .

As  $\nu^\wedge \langle (\ell^*, \delta) \rangle \in I_2$ , necessarily clauses (a)-(d) of  $\circledast$  hold with  $(\nu^\wedge \langle (\ell^*, \delta), (2, \zeta) \rangle, m)$  here standing for  $(\eta, n)$  there, recalling  $m = \ell g(\nu)$  so by clause (b) of  $\circledast$  we know that  $\text{cf}(\delta) \geq \theta_1$  but  $\partial = \text{cf}(\delta)$  hence  $\partial \geq \theta_1$ . Also  $\{\nu^\wedge \langle (\ell^*, \delta), (2, \alpha) \rangle : \alpha < \zeta\}$  is a subset of  $I_2$  and cofinal in  $J_{\bar{\eta}}$  and is increasing with  $\alpha$  so  $\text{cf}(J_{\bar{\eta}}) = \theta_2$  so clause (e) of  $\boxtimes_3$  applies.

As the two subcases 6A,6B are complimentary case 6 is done.

Finishing the proof of  $\boxtimes_3$ :

It is easy to check that our cases cover all the possibilities (as after discarding cases 0,1, if not case (6) then  $\nu^\wedge \langle (\ell^*, \delta) \rangle \notin I_2$ , as not case (3),  $\ell^* \neq 1$  but (see clause  $\odot(c)$  before case 2),  $\ell^* \in \{-2, 1\}$  so necessarily  $\ell^* = -2$ , so case (4),(5) cover the rest). Together we have proved  $\boxtimes_3$ .]

$\boxtimes_4$  recall  $\aleph_0 \leq \mu < \theta_1 < \theta_2$ ; if  $X \subseteq I_2, |X| < \theta_2$  then we can find  $Y$  such that  $X \subseteq Y \subseteq I_2, |Y| = \mu + |X|$ ,  $Y$  is unbounded in  $I_2$  from below and from above and for every  $\nu \in I_2 \setminus Y$  the following linear orders have cofinality  $\aleph_0$ :

(a)  $J_{Y, \nu}^2 = I_2 \upharpoonright \{\eta \in I_2 \setminus Y : (\forall \rho \in Y)(\rho <_{I_2} \nu \equiv \rho <_{I_2} \eta)\}$



- (b) the inverse of  $J_{Y,\nu}^2$
- (c)  $J_{Y,\nu}^- = I_2 \upharpoonright \{\eta \in I_2 : (\forall \rho \in J_{Y,\nu}^2)(\eta <_{I_2} \rho)\}$
- (d) the inverse of  $J_{Y,\nu}^+ := I_2 \upharpoonright \{\eta \in I_2 : (\forall \rho \in J_{Y,\nu}^2)(\rho <_{I_2} \eta)\}$ .

[Why? Let  $\mathcal{U} = \{\alpha^{\eta(\ell)} : \eta \in X \text{ and } \ell < \ell g(\eta)\}$ .

We choose  $W_n$  by induction on  $n < \omega$  such that

- $\Box_1$  (a)  $\mathcal{U} \subseteq W_n \subseteq \zeta + \omega$
- (b)  $W_n$  has cardinality  $\mu + |\mathcal{U}| = \mu + |X|$  and  $m < n \Rightarrow W_m \subseteq W_n$
- (c)  $\mu \subseteq W_0$  and  $\zeta + n \in W_0$  for  $n < \omega$
- (d)  $\alpha \in W_n \Rightarrow \alpha + 1 \in W_{n+1}$
- (e)  $\alpha + 1 \in W_n \Rightarrow \alpha \in W_{n+1}$
- (f) if  $\delta \in W_n$  is a limit ordinal of cofinality  $< \theta_1$  then  $\delta = \sup(\delta \cap W_{n+1})$
- (g) if  $\delta \in W_n$  and  $\text{cf}(\delta) \geq \theta_1$  (or just  $\text{cf}(\delta) \leq \mu + |X|$ ) then  $\sup(\delta \cap W_n) + 1 \in W_{n+1}$ .

This is straight. Let  $W = \cup\{W_n : n < \omega\}$ , so

- $\Box_2$   $\mathcal{U} \subseteq W$  and  $|W| = \mu + |X|$  and  $W$  satisfies
- (a)  $W \subseteq \zeta + \omega$
- (b)  $|W| < \theta_2$
- (c)  $0 \in W$  and  $\{\zeta + m : m < \omega\} \subseteq W$
- (d)  $\alpha \in W \Leftrightarrow \alpha + 1 \in W$
- (e) if  $\delta \in W$  and  $\aleph_0 < \text{cf}(\delta) \leq \mu$  then  $\delta = \sup(W \cap \delta)$
- (f) if  $\delta \in W$  and  $\text{cf}(\delta) \geq \theta_1$  or  $\text{cf}(\delta) = \aleph_0$  then  $\text{cf}(\text{otp}(W \cap \delta)) = \aleph_0$ .

Let  $Y = \{\eta \in I_2 : \alpha^{\eta(\ell)} \in W \text{ for every } \ell < \ell g(\eta)\}$ . Clearly  $X \subseteq Y$  and  $|Y| = \aleph_0 + |W| = \mu + |\mathcal{U}| < \theta_2$ . It suffices to check that  $Y$  is as required in  $\Box_4$ . From now on we shall use only the choice of  $Y$  and clauses (a)-(f) of  $\Box_2$ . By  $\Box_2(c)$  and the choice of  $Y$  clearly  $Y$  is unbounded in  $I_2$  from above and from below.

So let  $\nu \in I_2 \setminus Y$ , as  $\nu \upharpoonright 0 \in Y$  there is  $n < \ell g(\nu)$  such that  $\nu \upharpoonright n \in Y$ ,  $\nu \upharpoonright (n+1) \notin Y$ , so  $\alpha^{\nu(n)} < \zeta + \omega$ , and  $\alpha^{\nu(n)} \notin W$ , but by clause (c) of  $\Box_2$  we have  $\{\zeta + m : m < \omega\} \subseteq W$  hence  $\alpha^{\nu(n)} < \zeta$  and so  $\alpha_1 := \text{Min}(W \setminus \alpha^{\nu(n)})$  is well defined, is  $\leq \zeta$  and  $> \alpha^{\nu(n)}$ . As clearly  $0 \in W$ ,  $\beta \in W \Leftrightarrow \beta + 1 \in W$  by the choice of  $W$ , obviously  $\alpha_1$  is a limit ordinal. By clause (e) of  $\Box_2$  clearly  $\alpha_1$  is of cofinality  $\aleph_0$  or  $\geq \theta_1 = \mu^+$ . So clearly  $\alpha_0 := \sup(W \cap \alpha^{\nu(n)}) = \sup(W \cap \alpha_1) = \min\{\alpha : W \cap \alpha = W \cap \alpha^{\nu(n)}\}$

is a limit ordinal  $\leq \alpha^{\nu(n)}$  and  $\alpha_0 \notin W$  so  $\text{cf}(\alpha_0) \leq |W| < \theta_2$  but by the assumption on  $W$ , (see clause (f) of  $\square_2$ ) we have  $\text{cf}(\alpha_0) = \aleph_0$ . So  $(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_0) \rangle \in J_{Y,\nu}^2$ ; moreover

$\square_3$   $\rho \in J_{Y,\nu}^2$  iff  $\rho \in I_2$  satisfies one of the following:

- (a) (i)  $\nu \upharpoonright n = \rho \upharpoonright n$ , and  $\ell^{\nu(n)} = \ell^{\rho(n)}$ ,  
(ii)  $\alpha^{\rho(n)} \in [\alpha_0, \alpha_1)$
- (b) (i)  $\nu \upharpoonright n = \rho \upharpoonright n$ , and  $\ell^{\nu(n)} = \ell^{\rho(n)}$ ,  
(ii)  $\alpha^{\rho(n)} = \alpha_1$  and  $\alpha^{\rho(n+1)} \in [\sup(W \cap \zeta), \zeta)$   
(iii)  $(\ell^{\rho(n+1)}, \ell^{\rho(n)}) = (\ell^{\rho(n_1)}, \ell^{\nu(n)}) \in \{(2, -2), (2, 1), (-2, -1), (-2, 2)\}$
- (c) (i)  $\alpha_1 = \zeta$  and  $n > \theta$  and  $(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1) \rangle \notin I_2$   
(ii)  $(\ell^{\nu(n)}, \ell^{\nu(n-1)}) \in \{(2, -2), (2, 1), (-2, 2), (-2, -1)\}$   
(iii)  $\text{cf}(\nu(n)) \geq \theta_1$  and  $\nu(n) > \sup(W \cap \nu(n))$   
(iv)  $\rho \upharpoonright (n-1) = \nu \upharpoonright (n-1)$ ,  $\ell^{\rho(n-1)} = \ell^{\nu(n-1)}$   
(v)  $\alpha^{\rho(n-1)} \in [\sup(\nu(n-1) \cap W), \nu(n-1))$ .

[Why? First note that if  $\rho \in J_{Y,\nu}^2$  and  $\rho \upharpoonright k = \nu \upharpoonright k$ ,  $\rho(k) \neq \nu(k)$ , and  $k \leq n$  then necessarily  $k = n \wedge \ell^{\rho(k)} = \ell^{\nu(k)}$ . We now proceed to check “if”. Let  $f : \{-2, -1, 1, 2\} \rightarrow \{2, -2\}$  so that  $f^{-1}(\{2\}) = \{-2, 1\}$  and  $f^{-1}(\{-2\}) = \{-1, 2\}$ . Case (a) is obvious. In case (b) in order for  $\eta \in Y$  to separate between  $\nu$  and  $\rho$  it is necessary that  $\eta \upharpoonright (n+1) = \rho \upharpoonright (n+1)$ ,  $\ell^{\eta(n+1)} = \ell^{\rho(n+1)} = f(\ell^{\rho(n)})$  and that  $\alpha^{\eta(n+1)} \geq \zeta$ , but then  $\eta \notin I_2$ . In case (c) in order to separate between  $\rho$  and  $\nu$  by  $\eta \in Y$  there are two possibilities. Either  $\eta \upharpoonright n = \nu \upharpoonright n$  and then  $\ell^{\eta(n)} = \ell^{\nu(n)} = f(\ell^{\nu(n-1)})$  (recall that  $(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1) \rangle \notin I_2$ ), and  $\alpha^{\eta(n)} \geq \zeta$ , but then also  $\eta \notin I_2$ . The other possibility is that  $\eta \upharpoonright (n-1) = \nu \upharpoonright (n-1)$ ,  $\ell^{\eta(n-1)} = \ell^{\nu(n-1)}$  and  $\alpha = \alpha^{\eta(n-1)}$  is such that  $\alpha \in W$  and  $\alpha^{\rho(n-1)} < \alpha < \alpha^{\nu(n-1)}$  which is also impossible by the choice of  $\alpha^{\rho(n-1)}$ . Showing that these are the only cases (the “only if” direction) is similar and is actually done below.]

Now we proceed to check that clauses of  $\boxtimes_4$  hold.

Clause (a):

First assume  $\ell^{\nu(n)} \in \{-2, 1\}$ , and let  $J = \{\nu \upharpoonright n^\wedge \langle (\ell^{\nu(n)}, \alpha_0), (2, \zeta + m) : m < \omega \rangle\}$ . Now  $J \subseteq I_2$  [why? clearly if  $\rho \in J$  then  $\rho \upharpoonright (n+1) \in I_2$  so we only need to check  $\circledast$  for  $n$ , recall that  $\text{cf}(\alpha_0) = \aleph_0 < \theta_1$ , hence clause (b) of  $\circledast$  fails]. Now by clause (a) of  $\square_3$  we have that  $J \subseteq J_{Y,\nu}^2$ , and we claim that it is also cofinal in it. [Why? Note that as  $\ell^{\nu(n)} \in \{-2, 1\}$  then  $\nu \upharpoonright n^\wedge \langle (\ell^{\nu(n)}, \alpha_0) \rangle <_{I_2} \nu \upharpoonright (n+1)$ , and if  $\rho \in J_{Y,\nu}^2$  is as in clauses (a) or (b) of  $\square_3$  then for every  $m$  large enough  $\rho <_{I_2} \nu \upharpoonright$

$n^\wedge \langle (\ell^{\nu(n)}, \alpha_0), (2, \zeta + m) \rangle$ . If  $\rho \in J_{Y,\nu}^2$  is as in clause (c) of  $\Box_3$  then  $\ell^{\nu(n)} \in \{-2, 2\}$  by (ii) there, and as in this case  $\ell^{\nu(n)} \in \{-2, 1\}$ , necessarily  $\ell^{\nu(n)} = -2$  and so by (ii) of (c) of  $\Box_3$  we have  $\ell^{\nu(n-1)} \in \{-1, 2\}$ , but then  $\rho <_{I_2} \nu$  and so it is below every element in  $J$ .]

Second, assume  $\ell^{\nu(n)} \in \{-1, 2\}$  and  $\nu \upharpoonright n^\wedge \langle (\ell^{\nu(n)}, \alpha_1) \rangle \in I_2$ ; let  $\delta^* = \sup(W \cap \zeta)$ , so as above  $\delta^* \notin W$ , and has cofinality  $\aleph_0$  (which is less than  $\theta_1$ ), recall also that  $\text{cf}(\alpha_1) \geq \theta_1$ . So (for  $\ell \in \{-2, -1, 1, 2\}$ ) by  $\otimes$  we have  $(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1), (\ell, \beta) \rangle \in I_2$  iff  $\beta < \zeta \wedge \ell \in \{-2, -1, 1, 2\}$  or  $(\zeta \leq \beta < \zeta + \omega \wedge \ell \neq -2)$ . Hence we have  $(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1), (-2, \beta) \rangle \in I_2 \Leftrightarrow \beta < \zeta$ . Also  $(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1), (-2, \beta) \rangle \in Y \Leftrightarrow \beta \in W$ , and as  $\nu(n) < \alpha_1 \wedge \ell^{\nu(n)} \in \{-1, 2\}$  clearly  $\nu <_{I_2} (\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1), (-2, \beta) \rangle$ . Easily  $\{(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1), (-2, \varepsilon) \rangle : \varepsilon \in W \cap \zeta\}$  is a subset of  $\{\eta \in Y : \nu <_{I_2} \eta\}$  unbounded from below in it.

So  $\{(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1), (-2, \delta^*), (2, \alpha) \rangle : \zeta < \alpha < \zeta + \omega\}$  is included in  $I_2$  (recalling clause (b) of  $\otimes$  as  $\text{cf}(\delta^*) = \aleph_0$ ) and moreover is a cofinal subset of  $J_{Y,\nu}^2$  of order type  $\omega$ , so  $\text{cf}(J_{Y,\nu}^2) = \aleph_0$  as required.

Third, assume  $\rho^{\nu(n)} \in \{-1, 2\}$  and  $(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1) \rangle \in I_2$  and  $\text{cf}(\alpha_1) < \theta_1$ , equivalently  $\text{cf}(\alpha_1) = \aleph_0$  by clause (e) of  $\Box_2$ . In this case  $\{(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha) \rangle : \zeta \leq \beta < \zeta + \omega\}$  is included in  $I_2$  (recalling clause (b) of  $\otimes$ ) and in  $Y$ , hence recalling  $\Box_3(a)$  the set  $\{(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha) \rangle : \alpha \in [\alpha_0, \alpha_1]\}$  is a cofinal subset of  $J_{Y,\nu}^2$  hence its cofinality is  $\text{cf}(\alpha_1) = \aleph_0$  as required.

Fourth, we are left with the case  $\ell^{\nu(n)} \in \{-1, 2\}$  and  $(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1) \rangle \notin I_2$  so necessarily  $n > 0$  and clauses (a)-(d) of  $\otimes$  hold for it for  $n-1$ ; then by clause (c) of  $\otimes$  (recalling  $\alpha_1 \leq \zeta$  as shown before  $\Box_3$ ) necessarily  $\alpha_1 = \zeta$ . Clearly  $k := n-1 \geq 0$  and as clause (d) of  $\otimes$  holds and it says there “ $\ell^{\eta(n+1)} \in \{2, -2\}$ ” which means here  $\ell^{\nu(n)} \in \{2, -2\}$  but we are assuming presently  $\ell^{\nu(n)} \in \{-1, 2\}$  hence  $\ell^{\nu(n)} = \ell^{\nu(k+1)} = 2$  so using clause (d) of  $\otimes$ , see above, it follows that  $\ell^{\nu(k)} \in \{-2, 1\}$  and by clause (b) of  $\otimes$  we have  $\text{cf}(\alpha^{\nu(k)}) \geq \theta_1$ . Let  $\delta_* = \sup(W \cap \alpha^{\nu(k)})$ . Now if  $\delta_* < \alpha^{\nu(k)}$  then by clause (f) of  $\Box_2$  we know  $\text{cf}(\delta_*) = \aleph_0$  and  $\{(\nu \upharpoonright k)^\wedge \langle (\ell^{\nu(k)}, \delta_*) \rangle : m < \omega\}$  is included in  $I_2$  (as  $\nu \in I_2$  and  $\delta_* \leq \alpha^{\nu(k)}$  we have to check in  $\otimes$  only with  $k+1$  here standing for  $n$  there, but  $\text{cf}(\delta_*) = \aleph_0$  so clause (b) there fails) and so recalling  $\Box_3(c)$  this set is a cofinal subset of  $J_{Y,\nu}^2$  exemplifying that its cofinality is  $\aleph_0$ .

Lastly, if  $\delta_* = \alpha^{\nu(k)}$  then  $\langle (\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha) \rangle : \alpha \in W \cap \zeta \rangle$  is  $<_{I_2}$ -increasing with  $\alpha$ , all members in  $Y$ , and in  $J_{Y,\nu}^2$ , cofinal in it and has order type  $\text{otp}(W \cap \zeta)$  which has cofinality  $\aleph_0$  so also  $J_{Y,\nu}^2$  has cofinality  $\aleph_0$  as required.

Clause (b): What about the cofinality of the inverse? Recall that  $I_2$  is isomorphic to its inverse by the mapping  $(\ell, \beta) \mapsto (-\ell, \beta)$ , but this isomorphism maps  $Y$  onto itself hence it maps  $J_{Y,\nu}^2$  onto  $J_{Y,\nu'}^2$  for some  $\nu' \in I_2 \setminus Y$ , but clause (a) was proved also for  $\nu'$ , so this follows.

Clause (c): As  $Y$  is unbounded from below in  $I_2$  (containing  $\{(-2, \zeta + n) : n < \omega\}$ ) it follows that  $J_{Y,\nu}^-$  is non-empty, hence  $\text{cf}(J_{Y,\nu}^-) \neq 0$ , but what is  $\text{cf}(J_{Y,\nu}^-)$ ?

First, if  $\ell^{\nu(n)} \in \{-1, 2\}$  then  $\{(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha) \rangle : \alpha < \alpha_0\}$  is an unbounded subset of  $J_{Y,\nu}^-$  of order type  $\alpha_0$  hence  $\text{cf}(J_{Y,\nu}^-) = \text{cf}(\alpha_0) = \aleph_0$  (see the assumption on  $W$  and the choice of  $\alpha_0$ ).

Second, if  $\ell^{\nu(n)} = \{-2, 1\}$  and  $(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1) \rangle \in I_2$  and  $\text{cf}(\alpha_1) \geq \theta_1$  then as in the proof of clause (a) we have  $\{(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1), (2, \zeta + m) \rangle \notin I_2 \text{ for } m < \omega$  and again letting  $\delta^* = \sup(W \cap \zeta)$  we have  $\{(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1), (2, \beta) \rangle : \beta \in W \cap \zeta\}$  is included in  $I_2$  and in  $J_{Y,\nu}^-$  and even is an unbounded subset of  $J_{Y,\nu}^-$  of order type  $\text{otp}(W \cap \delta^*)$  which has the same cofinality as  $\delta^*$  which is  $\aleph_0$ .

Third, if  $\ell^{\nu(n)} \in \{-2, 1\}$  and  $(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1) \rangle \in I_2$  and  $\text{cf}(\alpha_1) < \theta_1$ , equivalently  $\text{cf}(\alpha_1) = \aleph_0$ , then  $\{(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1), (2, \zeta + m) \rangle : m < \omega\}$  is a subset of  $I_2$  (as  $\text{cf}(\alpha_1) = \aleph_0$ ) is included in  $J_{Y,\nu}^-$ , unbounded in it and has cofinality  $\aleph_0$ , so we are done.

Fourth and lastly, if  $\ell^{\nu(n)} \in \{-2, 1\}$  and  $(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \alpha_1) \rangle \notin I_2$  then as in the proof of clause (a) we have  $\alpha_1 = \zeta$  and again letting  $\delta^* = \sup(W \cap \zeta)$  we have  $\text{cf}(\delta^*) = \aleph_0$  and  $(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \delta^*) \rangle \in I_2$  and  $\{(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \delta^*), (2, \zeta + m) \rangle : m < \omega\}$  is a subset of  $I_2$ , moreover a subset of  $J_{Y,\nu}^-$  unbounded in it and  $(\nu \upharpoonright n)^\wedge \langle (\ell^{\nu(n)}, \delta^*), (2, \zeta + m) \rangle$  is  $<_{I_2}$ -increasing with  $m$ . So indeed  $J_{Y,\nu}^-$  has cofinality  $\aleph_0$ .

Clause (d): As in clause (b) we use the anti-isomorphism.  
So  $\boxtimes_4$  holds.]

$\boxtimes_5$  if  $I' \subseteq I_2$  then the number of cuts of  $I'$  induced by members of  $I_2 \setminus I'$ , that is  $\{ \{s \in I' : s <_{I_2} t\} : t \in I_2 \setminus I' \}$  is  $\leq |I'| + 1$ .  
[Why? Let  $\mathcal{U} := \{\alpha^{\eta(\ell)} : \ell < \ell g(\eta) \text{ and } \eta \in I'\}$ , it belongs to  $[\zeta + \omega]^{\leq \mu}$ . Now (by inspection)  $\eta_1, \eta_2 \in I_2 \setminus I'$  realizes the same cut of  $I'$  when:

- (a)  $\ell g(\eta_1) = \ell g(\eta_2)$
- (b)  $\ell^{\eta_1(n)} = \ell^{\eta_2(n)}$  for  $n < \ell g(\eta_1)$
- (c)  $\alpha^{\eta_1(n)} \in \mathcal{U} \Leftrightarrow \alpha^{\eta_2(n)} \in \mathcal{U} \Rightarrow \alpha^{\eta_1(n)} = \alpha^{\eta_2(n)}$  for  $n < \omega$
- (d)  $\beta < \alpha^{\eta_1(n)} \equiv \beta < \alpha^{\eta_2(n)}$  for  $\beta \in \mathcal{U}$  and  $n < \omega$

[Why? Now clauses (a)-(d) define an equivalence relation on  $I_2 \setminus I'$  which refines “inducing the same cut” and has  $\leq |\mathcal{U}| + \aleph_0 = |I'| + \aleph_0$  equivalence classes. As the case  $I'$  is finite is trivial, we are done proving  $\boxtimes_5$ .]

$\boxtimes_6$  if  $\partial$  is regular uncountable,  $n^* < \omega$  and  $t_{\varepsilon, \ell} \in I_2$  for  $\varepsilon < \partial, \ell < n^*$  and  $t_{\varepsilon, 0} <_{I_2} \dots <_{I_2} t_{\varepsilon, n^*-1}$  for  $\varepsilon < \partial$  then for some unbounded (and even

stationary) set  $S \subseteq \partial$ ,  $m \leq n^*$  and  $0 = k_0 < k_1 < \dots < k_m = n^*$  stipulating  $t_{\varepsilon, k_m} = \infty$  and letting  $\varepsilon(*) = \text{Min}(S)$  we have:

- (a) for each  $i < m$ :
  - ( $\alpha$ ) if  $\varepsilon < \xi$  are from  $S$  and  $\ell_1, \ell_2 \in [k_i, k_{i+1})$  then  $t_{\varepsilon, \ell_1} <_{I_2} t_{\xi, \ell_2}$  or
  - ( $\beta$ ) if  $\varepsilon < \xi$  are from  $S$  and  $\ell_1, \ell_2 \in [k_i, k_{i+1})$  then  $t_{\xi, \ell_2} <_{I_2} t_{\varepsilon, \ell_1}$  or
  - ( $\gamma$ )  $k_{i+1} = k_i + 1$  and for every  $\varepsilon \in S$  we have  $t_{\varepsilon, k_i} = t_{\varepsilon(*), k_i}$
- (b) there is a sequence  $\langle s_i^-, s_i^+ : i < m \rangle$  such that
  - ( $\alpha$ )  $i < m \Rightarrow s_i^- <_{I_2} s_i^+$
  - ( $\beta$ ) if  $i < m - 1$  then  $s_i^+ < s_{i+1}^-$  except possibly when  $\langle t_{\varepsilon, k_i} : \varepsilon < \partial \rangle$  is  $<_{I_2}$ -decreasing and there is no  $t \in I_2$  such that  $\varepsilon < \partial \Rightarrow t_{\varepsilon, k_i} <_{I_2} t <_{I_2} t_{\varepsilon, k_{i+1}}$ , hence (by  $\boxtimes_3$ ) we have  $\partial \geq \theta_2$
  - ( $\gamma$ ) for each  $i < m$  the set  $\{t_{\varepsilon, \ell} : \varepsilon \in S \text{ and } \ell \in [k_i, k_{i+1})\}$  is included in the interval  $(s_i^-, s_i^+)_{I_2}$ .

[Why? Straight. For some stationary  $S_1 \subseteq \partial$  and  $\langle n_k : k < n^* \rangle$  we have  $\varepsilon \in S_1 \wedge k < n^* \Rightarrow \ell g(t_{\varepsilon, k}) = n_k$ . Without loss of generality also  $\langle \ell^{t_{\varepsilon, k}(i)} : i < n_k \rangle$  does not depend on  $\varepsilon \in S_1$ . By  $\sum_{k < n^*} n_k$  application of  $\partial \rightarrow (\partial, \omega)^2$ ,

without loss of generality for each  $k < n^*$  and  $i < n_k$  the sequence  $\langle \alpha^{t_{\varepsilon, k}(i)} : \varepsilon \in S_1 \rangle$  is constant or increasing. Cleaning a little more we are done.

So  $\boxtimes_6$  holds.]

Lastly, recall that we chose  $I$  to be  $(|M|, <^M)$ , where  $M$  was the real closure of  $M_0$  and (see  $(*)_1$ ),  $M_0$  the ordered field generated over  $\mathbb{Q}$  by  $\{a_t : t \in I_2\}$  as described in  $(*)_1$  above and for every  $u \subseteq \zeta$  let:

- (\*)<sub>2</sub> (a)  $I_u^1 = \{(\ell, \beta) \in I_1 : \beta \in u \text{ or } \beta \in [\zeta, \zeta + \omega)\}$
- (b)  $I_u^{*,2} = \{\eta \in I_2^* : \alpha^{\eta(\ell)} \in I_u^1 \text{ for every } \ell < \ell g(\eta)\}$
- (c)  $I_u^2 = \{\eta \in I_2 : \alpha^{\eta(\ell)} \in I_u^1 \text{ for every } \ell < \ell g(\eta)\}$
- (d)  $I_u$  = the real closure of  $\mathbb{Q}(a_t : t \in I_u^2)$  in  $M$
- (e) for  $t \in I_2 \setminus I_u^2$  let
 
$$I_{u,t}^2 = I_2 \upharpoonright \{s \in I_2 : s \notin I_u^2 \text{ and for every } r \in I_u^2 \text{ we have } r <_{I_2} t \equiv r <_{I_2} s\}$$
- (f) for  $x \in I \setminus I_u$  let
 
$$I_{u,x} = I \upharpoonright \{y \in I : y \notin I_u \text{ and } (\forall a \in I_u)(a <_I y \equiv a <_I x)\}$$
- (g) let  $\hat{I}_u$  be the set  $I_u \cup \{I_{u,a} : a \in I \setminus I_u\}$  ordered by:  $x <_{\hat{I}_u} y$  iff

one of the following holds:

- ( $\alpha$ )  $x, y \in I_u$  and  $x <_{I_u} y$
- ( $\beta$ )  $x \in I_u, y = I_{u,b}$  and  $x <_{I_u} b$
- ( $\gamma$ )  $x = I_{u,a}, y \in I_u$  and  $a <_{I_u} y$
- ( $\delta$ )  $x = I_{u,a}, y = I_{u,b}$  and  $a <_{I_u} b$  (can use it more!)  
(note that by  $\boxtimes_5$ ,  $|u| \leq \mu \Rightarrow |\hat{I}_u| \leq \mu$ ).

Now observe

- ( $\ast$ )<sub>3</sub> for  $u \subseteq \zeta$ ,  $I_u^2$  is unbounded in  $I_2$  from below and from above.

We define

- ( $\ast$ )<sub>4</sub> we say<sup>4</sup> that  $u$  is  $\mu$ -reasonable if:
  - (a)  $u \subseteq \zeta, |u| < \theta_2$  and  $\mu \subseteq u$
  - (b)  $\alpha \in u \equiv \alpha + 1 \in u$  for every  $\alpha$
  - (c) if  $\delta \in u$  and  $\aleph_0 \leq \text{cf}(\delta) \leq \mu$  then  $\delta = \sup(u \cap \delta)$
  - (d) if  $\delta \leq \zeta$  and  $\text{cf}(\delta) > \mu$  then  $\text{cf}(\text{otp}(\delta \cap u)) = \aleph_0$ .

Now we note

- ( $\ast$ )<sub>5</sub> if  $X \subseteq I$  has cardinality  $< \theta_2$  and  $u_* \subseteq \zeta$  has cardinality  $< \theta_2$  then we can find a  $\mu$ -reasonable  $u$  such that  $X \subseteq I_u$  and  $u_* \subseteq u$  and  $|u| = \mu + |X| + |u_*|$ .

[Why? By the proof of  $\boxtimes_4$ .]

- ( $\ast$ )<sub>6</sub> if  $u$  is  $\mu$ -reasonable then  $Y := I_u^2$  satisfies the conclusions of  $\boxtimes_4$ .

[Why? By the proof of  $\boxtimes_4$ , that is if  $u^+ := u \cup \{\zeta + n : n < \omega\}$  then  $Y$  as defined in the proof there using  $u^+$  for  $W$ , is  $I_u^2$  from ( $\ast$ )<sub>2</sub>(c), and it satisfies demands (a)-(f) from  $\boxtimes_2$  so the proof there applies.]

- ( $\ast$ )<sub>7</sub> if  $u$  is  $\mu$ -reasonable and  $x \in I \setminus I_u$  then  $\text{cf}(I_{u,x}) \leq \aleph_0$ .

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<sup>4</sup>we may in clauses (e) + (c) replace  $\mu$  by  $\mu + |\mathcal{U}|$ , no harm and it makes (c)( $\beta$ ) of ( $\ast$ )<sub>1</sub>, redundant

Why? The proof takes awhile. Toward contradiction assume  $\partial = \text{cf}(I_{u,x})$  is  $> \aleph_0$  and let  $\langle b_\varepsilon : \varepsilon < \partial \rangle$  be an increasing sequence of members of  $I_{u,x}$  unbounded in it. So for each  $\varepsilon < \partial$  there is a definable function  $f_\varepsilon(x_0, \dots, x_{n(\varepsilon)-1})$  where definable of course means in the theory of real closed fields and  $t_{\varepsilon,0} <_{I_2} t_{\varepsilon,1} <_{I_2} \dots <_{I_2} t_{\varepsilon,n(\varepsilon)-1}$  from  $I_2$  such that  $M \models "b_\varepsilon = f_\varepsilon(a_{t_{\varepsilon,0}}, \dots, a_{t_{\varepsilon,n(\varepsilon)-1}})"$  and  $n(\varepsilon)$  is minimal. As  $\text{Th}(\mathbb{R})$  is countable and  $\aleph_0 < \partial = \text{cf}(\partial)$ , without loss of generality  $\varepsilon < \partial \Rightarrow f_\varepsilon = f_*$  so  $\varepsilon < \partial \Rightarrow n(\varepsilon) = n(*)$ .

Apply  $\boxtimes_6$  to  $\langle \bar{t}^\varepsilon = \langle t_{\varepsilon,\ell} : \ell < n(*) \rangle : \varepsilon < \partial \rangle$  and get  $S \subseteq \partial$  and  $0 = k_0 < k_1 < \dots < k_m = n(*)$  and  $\langle (s_i^-, s_i^+) : i < m \rangle$  and  $\varepsilon(*) = \text{Min}(S)$  as there. Without loss of generality the truth value of " $t_{\varepsilon,\ell} \in I_u^2$ " for  $\varepsilon \in S$ , depends just on  $\ell$ . Let  $w_1 = \{i < m : (\forall \varepsilon \in S)(t_{\varepsilon,k_i} = t_{\varepsilon(*),k_i})\}$ ,  $w_2 = \{\ell < n(*) : t_{\varepsilon(*),\ell} \in I_u^2\}$ ; clearly for every  $\ell < n(*)$  we have  $(\forall \varepsilon \in S)(t_{\varepsilon,\ell} = t_{\varepsilon(*),\ell}) \Leftrightarrow \ell \in \{k_i : i \in w_1\}$  and  $i \in w_1 \Rightarrow k_i + 1 = k_{i+1}$ .

Let  $t_{k_i}^* = t_{\varepsilon(*),k_i}$  for  $(\varepsilon < \partial \text{ and } i \in w_1)$ . Renaming without loss of generality  $S = \partial$  and  $\varepsilon(*) = 0$ .

We have some free choice in choosing  $\langle b_\varepsilon : \varepsilon < \partial \rangle$  (as long as it is cofinal in  $I_{u,x}$ ), so without loss of generality we choose it such that  $n(*)$  is minimal and then  $|w_1|$  is maximal and then  $|w_2|$  is maximal.

Now does the exceptional case in (b)( $\beta$ ) of  $\boxtimes_6$  occurs? This is an easier case and we delay it to the end.

As  $I_2$  and  $I_{2,<t}$  for  $t \in I_2$  have cofinality  $\aleph_0$  (see  $\boxtimes_2(a)$ , (b)) and  $\boxtimes_3$  and this holds for the inverse of  $I_2$ , too, while  $\partial = \text{cf}(\partial) > \aleph_0$  and we can replace  $\langle b_\varepsilon : \varepsilon < \partial \rangle$  by  $\langle b_{n(*)+\varepsilon} : \varepsilon < \partial \rangle$  we can find  $t_{\partial,\ell}$  ( $\ell < n(*)$ ) such that

- ⊙ (a)  $t_{\partial,0} <_{I_2} t_{\partial,1} <_{I_2} \dots <_{I_2} t_{\partial,n(*)-1}$
- (b) if  $\varepsilon < \xi < \partial$  and  $\ell_1, \ell_2 < n(*)$  then  $(t_{\varepsilon,\ell_1} <_{I_2} t_{\partial,\ell_2}) \equiv (t_{\varepsilon,\ell_1} <_{I_2} t_{\xi,\ell_2})$   
and  $(t_{\partial,\ell_1} <_{I_2} t_{\varepsilon,\ell_2}) \equiv (t_{\xi,\ell_1} <_{I_2} t_{\varepsilon,\ell_2})$
- (c) if  $\ell \in [k_i, k_{i+1})$  then  $t_{\partial,\ell} \in (s_i^-, s_i^+)_{I_2}$ .

Case 0:  $\{0, \dots, m-1\} = w_1$ .

This implies  $i < m \Rightarrow k_i + 1 = k_{i+1}$  hence  $m = n$  hence  $\ell < n \Rightarrow t_{\xi,\ell} = t_\ell^*$  and so contradicts " $\langle b_\varepsilon : \varepsilon < \partial \rangle$  is increasing" (as it becomes constant).

Case 1:  $[0, m) \setminus w_1$  is not a singleton.

It cannot be empty by "not case 1". Choose  $i(*) \in \{0, \dots, m-1\} \setminus w_1$  and for  $\varepsilon, \xi < \partial$  let  $\bar{t}^{\varepsilon,\xi} = \langle t_\ell^{\varepsilon,\xi} : \ell < n(*) \rangle$  be defined by:  $t_\ell^{\varepsilon,\xi}$  is  $t_{\varepsilon,\ell}$  if  $\ell \in [k_{i(*)}, k_{i(*)+1})$  and  $t_{\xi,\ell}$  otherwise. Let  $b_{\varepsilon,\xi} = f_*(a_{t_0^{\varepsilon,\xi}}, \dots, a_{t_{n(*)-1}^{\varepsilon,\xi}}) \in M$ .

Clearly

⊗<sub>0</sub> for any  $\varepsilon_1, \varepsilon_2, \xi_1, \xi_2 \leq \partial$  the truth value of  $b_{\varepsilon_1, \xi_1} < b_{\varepsilon_2, \xi_2}$  depend just on the inequalities which  $\langle \varepsilon_1, \varepsilon_2, \xi_1, \xi_2 \rangle$  satisfies and even just on the inequalities which the  $t_{\varepsilon_1, \ell}, t_{\varepsilon_2, \ell}, t_{\xi_1, \ell}, t_{\xi_2, \ell}$  ( $\ell < n(*)$ ) satisfy.

[Why? Recall  $\langle \langle t_{\varepsilon, \ell} : \ell < n(*) \rangle : \varepsilon \in S \rangle$  is an indiscernible sequence in the linear order  $I_2$  (for quantifier free formulas) and  $M$  has elimination of quantifiers.]

$$\otimes_1 \bigwedge_{\ell=1,2} \varepsilon(0) < \varepsilon_\ell < \varepsilon(1) < \partial \Rightarrow b_{\varepsilon(0)} <_I b_{\varepsilon_1, \varepsilon_2} <_I b_{\varepsilon(1)}.$$

[Why? By ⊗<sub>0</sub> the desire statement,  $b_{\varepsilon(0)} <_I b_{\varepsilon_1, \varepsilon_2} <_I b_{\varepsilon(1)}$  is equivalent to  $b_{\varepsilon(0)} < b_{\varepsilon_1, \varepsilon_1} < b_{\varepsilon(1)}$  which means  $b_{\varepsilon(0)} < b_{\varepsilon_1} < b_{\varepsilon(1)}$  which holds.]

$$\otimes_2 b_{0,2} <_I b_1.$$

[Why? Otherwise  $b_1 \leq_I b_{0,2}$  hence  $\varepsilon \in (0, \partial) \Rightarrow b_\varepsilon <_I b_{0, \varepsilon+1} <_I b_{\varepsilon+2}$  (by ⊗<sub>0</sub> + ⊗<sub>1</sub>) so  $\langle b_{0, \varepsilon} : \varepsilon \in (1, \partial) \rangle$  is also an increasing sequence unbounded in  $I_{u,x}$  contradiction to “ $w_1$  maximal”.]

$$\otimes_3 b_{0,2} < b_{1,2}.$$

[Why? By ⊗<sub>0</sub> + ⊗<sub>2</sub> we have  $b_{0,4} < b_1$  and by ⊗<sub>1</sub> we have  $b_1 < b_{2,4}$  together  $b_{0,4} < b_{2,4}$  so by ⊗<sub>0</sub> we have  $b_{0,2} < b_{1,2}$ .]

But then  $\langle b_{\varepsilon, \partial} : \varepsilon < \partial \rangle$  increases (by ⊗<sub>3</sub> + ⊗<sub>0</sub>) and  $\varepsilon < \partial \Rightarrow b_\varepsilon = b_{\varepsilon, \varepsilon} < b_{\varepsilon+1, \partial} < b_{\varepsilon+2}$  (by ⊗<sub>1</sub> and ⊗<sub>2</sub> respectively) hence is an unbounded subset of  $I_{u,x}$  contradiction to the maximality of  $|w_1|$ .

Case 2:  $m \setminus w_1 = \{0, \dots, m-1\} \setminus w_1$  is  $\{i(*)\}$ .

Subcase 2A: For some  $i < m, i \neq i(*)$  and  $j := k_i \notin w_2$ .

Choose such  $i$  with  $|i - i(*)|$  maximal. For any  $s$  let  $t_{\varepsilon, \ell, s}$  be  $t_{\varepsilon, \ell}$  if  $\ell \neq j$  and be  $s$  if  $\ell = j$ .

Let  $I' = \{s \in I_{u, t_{\varepsilon(*)}, j}^2 : s, t_{\varepsilon(*)}, j \text{ realize the same cut of } \{t_{\varepsilon, \ell} : \varepsilon < \partial, \ell \neq j\}\}$ , note that  $k_{j+1} = k_j + 1$ . Recalling  $\boxtimes_2(b)$ , the cofinality of  $I_{2, < t_{\varepsilon(*)}, j}$  is  $\aleph_0$  and also the cofinality of the inverse of  $I_{2, > t_{\varepsilon(*)}, j}$  is  $\aleph_0$  recalling the choice of  $\langle (s_\iota^-, s_\iota^+) : \iota < m \rangle$  there is an open interval<sup>5</sup> of  $I_2$  around  $t_{\varepsilon(*)}, j$  which is  $\subseteq I'$ . Note that  $I'$  is dense in itself and has neither first nor last member by  $\boxtimes_2 + \boxtimes_4(a), (b)$ .

As  $f_*$  is definable, by the choice of  $M_0$  and  $M$  and of  $I' \subseteq I_{u, t_{\varepsilon(*)}, j}^2$  we have: if  $\varepsilon < \partial \wedge s \in I'$  then  $t_{\varepsilon(*)}, j$  and  $s$  realize the same cut of  $I_u^2 \cup \{t_{\varepsilon, \ell} : \varepsilon < \partial, j \neq \ell\}$  hence  $f_*^M(\dots, a_{t_{\varepsilon, \ell}, s}, \dots)_{\ell < n}, b_\varepsilon$  realize the same cut of  $I_u$  which means that

<sup>5</sup>if we allow  $+\infty, -\infty$  as end points



$f_*(\dots, a_{t_{\varepsilon, \ell, s}}, \dots)_{\ell < n} \in I_{u, x}$  hence by the choice of  $\langle b_\varepsilon : \varepsilon < \partial \rangle$  we have  $(\exists \xi < \partial)(f_*(\dots, a_{t_{\varepsilon, \ell, s}}, \dots) < b_\xi)$ .

So again by the definability (and indiscernibility)

$$\textcircled{*}_4 \quad \varepsilon < \partial \wedge s \in I' \Rightarrow f_*^M(\dots, a_{t_{\varepsilon, \ell, s}}, \dots) < b_{\varepsilon+1}.$$

As  $I'$  is dense in itself, what we say on the pair  $(s, t_{\varepsilon(*), j})$  when  $s \in I' \wedge s <_{I_2} t_{\varepsilon(*), j}$  holds for the pair  $(t_{\varepsilon(*), j}, s)$  when  $s \in I' \wedge t_{\varepsilon(*), j} <_I s$  so

$$\textcircled{*}_5 \quad \varepsilon < \partial \wedge s \in I' \Rightarrow b_\varepsilon < f_*^M(\dots, a_{t_{\varepsilon+1, \ell, s}}, \dots)$$

(more fully let  $s_1 <_{I_2} t_{\varepsilon(*), j} <_{I_2} s_2$  and  $s_1, s_2 \in I'$  then the sequences  $\langle t_{\varepsilon, \ell} : \ell \neq j, \ell < n(*) \rangle \wedge \langle s_1 \rangle \wedge \langle t_{\varepsilon+1, \ell} : \ell \neq j, \ell < n(*) \rangle \wedge \langle t_{\varepsilon(*), j} \rangle$  and  $\langle t_{\varepsilon, \ell} : \ell \neq j, \ell < n(*) \rangle \wedge \langle t_{\varepsilon(*), j} \rangle \wedge \langle t_{\varepsilon+1, \ell} : \ell \neq j, \ell < n(*) \rangle \wedge \langle s_2 \rangle$  realizes the same quantifier free type in  $I_2$ , (recalling  $t_{\varepsilon, j} = t_{\varepsilon(*), j}$ ).

By  $\textcircled{*}_4 + \textcircled{*}_5$  and indiscernibility we can replace  $t_{\varepsilon(*), j}$  by any  $t' \in I'$  which realizes the same cut as  $t_{\varepsilon(*), j}$  of  $\{t_{\varepsilon, \ell} : \varepsilon < \partial, \ell \neq j\}$ . But if  $j > i(*)$  then  $\{t_{j+1}^*, \dots, t_{n(*)-1}^*\} \subseteq I_u^2$  by the choice of  $j$ , and the set  $I'' = \{t \in I_2 : \text{if } \varepsilon < \partial, \ell \neq j \text{ then } t \neq t_{\varepsilon, \ell} \text{ and } t_{\varepsilon, \ell} <_{I_2} t \equiv t_{\varepsilon, \ell} <_{I_2} t_j^*\}$  include an initial segment of  $J_{I_u^2, t_{\varepsilon(*), j}}^+$ , see  $\boxtimes_4(d)$ , i.e.  $(*)_6$  so its inverse has cofinality  $\aleph_0$ , say  $\langle s_n^* : n < \omega \rangle$  exemplifies this, so  $n < \omega \Rightarrow s_{n+1}^* <_{I_2} s_n^*$ . So for every  $\varepsilon < \partial$  for some  $n < \omega$ ,  $f_*^M(\dots, a_{t_{\varepsilon+1, \ell, s_n^*}}, \dots) \in (b_\varepsilon, b_{\varepsilon+1})_I$ . So for some  $n_* < \omega$  this holds for unboundedly many  $\varepsilon < \partial$ , contradictory to “ $|w_2|$  is maximal”. Similarly if  $j < i(*)$ .

Subcase 2B: For every  $\varepsilon < \partial$  for some  $\xi \in (\varepsilon, \partial)$ , the interval of  $I_2$  which is defined by  $t_{\varepsilon, k_{i(*)}}, t_{\xi, k_{i(*)}}$  is not disjoint to  $I_u^2$  [so without loss of generality has  $\geq k_{i(*)+1} - k_{i(*)}$  members of  $I_u^2$ ].

In this case as in case 1, without loss of generality  $\{k_{i(*)}, \dots, k_{i(*)+1}\} \subseteq w_2$  so as  $|w_2|$  is maximal this holds. So as not subcase 2A,  $\{t_{\varepsilon, \ell} : \varepsilon < \partial, \ell < n\} \subseteq I_u^2$  hence  $\{b_\varepsilon : \varepsilon < \partial\} \subseteq I_u$ , contradiction.

Subcase 2C: None of the above.

As not subcase(2B), without loss of generality  $\{t_{\varepsilon, \ell} : \varepsilon < \partial \text{ and } \ell \in [k_{i(*)}, k_{i(*)+1})\} \subseteq I_{u, t_{\varepsilon(*), k_{i(*)}}}^2$ . Then as in subcase(2A) the sequence  $\langle t_{\varepsilon, k_{i(*)}} : \varepsilon < \partial \rangle$  is increasing/decreasing and is unbounded from above/below in  $I_{u, t_{\varepsilon(*), k_{i(*)}}}^2$  contradiction to  $(*)_6$ .

In more detail, so  $I' := I_{u, t_{0, k_{i(*)}}}^2$  includes all  $\{t_{\varepsilon, \ell} : \varepsilon < \partial \text{ and } \ell \in [k_{i(*)}, k_{i(*)+1})\}$ . Also  $I'$  and its inverse are of cofinality  $\aleph_0$  by  $(*)_6$  hence without loss of generality we can find (new)  $\langle t_{\partial, \ell} : \ell \in [k_{i(*)}, k_{i(*)+1}) \rangle$  such that  $t_{\partial, \ell} <_{I_2} t_{\partial, \ell+1}$ ,  $t_{\partial, \ell} \in (s_{i(*)}^-, s_{i(*)}^+)_{I_2}$  and  $\varepsilon < \partial \Rightarrow t_{\varepsilon, \ell_1} <_{I_2} t_{\partial, \ell} \equiv t_{\varepsilon, \ell_1} < t_{\varepsilon+1, \ell_2}$  and the convex hull in  $I_2$  of  $\{t_{\zeta, \ell} : \zeta \leq \partial \text{ and } \ell \in [k_{i(*)}, k_{i(*)+1})\}$  is disjoint to  $I_u^2$ . Let  $t_{\partial, \ell} = t_{\partial, \ell}$  for  $\ell \notin [k_{i(*)}, k_{i(*)+1})$ ,  $\ell < m$ ,  $b_\partial = f_*(a_{t_{\partial, 0}}, \dots, a_{t_{\partial, n-1}})$ .

Easily  $\varepsilon < \partial \Rightarrow b_\varepsilon <_I b_\partial$ . As  $\varepsilon < \xi < \partial \Rightarrow (b_\varepsilon, b_\xi)_{I_2} \cap u = \emptyset$  easily  $\varepsilon < \partial \Rightarrow (b_\varepsilon, b_\partial)_{I_2} \cap u = 0$ , contradiction to  $\langle b_\varepsilon : \varepsilon < \partial \rangle$  being cofinal in  $I_{u,x}$ .

To finish proving  $(*)_7$ , we have to consider the possibility that applying  $\boxtimes_6$ , the exceptional case in  $(b)(\beta)$  of  $\boxtimes_6$  occurs for some  $i < m$  say for  $i(*)$ ; see before  $\odot$ .

Also without loss of generality as  $\partial \geq \theta_2$  then without loss of generality  $\ell \in w_2 \Rightarrow t_{\varepsilon,\ell} = t_{\varepsilon(*),\ell}$  and for each  $\ell < n(*)$  we have  $(\forall \varepsilon, \zeta < \partial)(\forall s \in I_u^2)(s <_{I_2} t_{\varepsilon,\ell} \equiv s <_{I_2} t_{\zeta,\ell})$ .

Now we can define  $\bar{t}^{\varepsilon,\xi} = \langle t_\ell^{\varepsilon,\xi} : \ell < n(*) \rangle$  as in case 1 and prove  $\otimes_0 - \otimes_3$  there.

Clearly all members of  $\{t_{\varepsilon,\ell} : \varepsilon < \partial, \ell \in [k_{i(*)}, k_{i(*)+2})\}$  realize the same cut of  $I_u^2$  and we get easy contradiction.

As we can use only  $\langle t_{n(*)}, \varepsilon : \varepsilon < \partial \rangle$  and add to  $f_*$  dummy variables, without loss of generality  $k_{i(*)+1} - k_{i(*)} = k_{i(*)+2} - k_{i(*)+1}$ . Let  $J$  be  $\{1, -1\} \times \partial$  ordered by  $(\ell_1, \varepsilon_1) <_J (\ell_2, \varepsilon_2)$  iff  $\ell_1 = 1 \wedge \ell_2 = -1$  or  $\ell_1 = 1 = \ell_2 \wedge \varepsilon_1 < \varepsilon_2$  or  $\ell_1 = -1 = \ell_2 \wedge \varepsilon_1 > \varepsilon_2$ .

For  $\iota \in J$  let  $\iota = (\ell^\iota, \varepsilon^\iota) = (\ell[\iota], \varepsilon[\iota])$ . For  $\zeta < \partial$  and  $\iota_1, \iota_2 \in J$  we define  $\bar{t}_{\zeta, \iota_1, \iota_2} = \langle t_{\zeta, \iota_1, \iota_2, n} : n < n(*) \rangle$  by  $t_{\zeta, \iota_1, \iota_2, n}$  is  $t_{\varepsilon[\iota_1], n}$  if  $n \in [k_{i(*)}, k_{i(*)+1})$ ,  $t_{\varepsilon[\iota_2], n}$  if  $n \in [k_{i(*)+1}, k_{i(*)+2})$  and  $t_{\zeta, n}$  otherwise. Now letting  $b_{\zeta, \iota_1, \iota_2} := f_*(\bar{t}_{\zeta, \iota_1, \iota_2})$

$\otimes_6$  all  $b_{\zeta, \iota_1, \iota_2}$  realize the same cut of  $I_u^2$ .

Now

$\otimes_7$  indiscernibility as in  $\otimes_0$  holds

$\otimes_8 \neg(b_{\zeta, (1, \varepsilon), (1, \varepsilon+1)} \leq_{I_*} b_{\zeta, (1, \varepsilon+2), (1, \varepsilon+3)})$ .

[Why? Otherwise by indiscernibility, if  $\zeta \in (6, \partial)$  then  $b_{\zeta, (1, \zeta), (-1, 3)} <_I b_{\zeta, (-1, 5), (-1, 4)}$ . Hence  $\langle b_{\zeta, (-1, 5), (-1, 4)} : \zeta \in (6, \partial) \rangle$  is monotonic in  $I_*$ , all members realizing the fix cut of  $I_u^2$  and is unbounded in it (by the inequality above) so contradiction to maximality of  $|w_j|$ .]

$\otimes_9 \neg(b_{\zeta, (1, \varepsilon+2), (1, \varepsilon+3)} <_I b_{\zeta, (1, \varepsilon), (1, \varepsilon+1)})$ .

[Similarly, as otherwise if  $\zeta \in (6, \partial)$  then  $b_{\zeta, (1, \zeta), (-1, \zeta)} <_I b_{\zeta, (1, 4), (1, 5)}$ . Hence  $\langle b_{\zeta, (1, 4), (1, 5)} : \zeta \in (6, \partial) \rangle$  contradict the maximality of  $(w_1)$ .]

So we have proved  $(*)_7$

$(*)_8$  if  $u$  is  $\mu$ -reasonable,  $x \in I \setminus I_u$  then  $\text{cf}(I_{u,x}) = \aleph_0$ .

[Otherwise by  $(*)_7$  it has a last element say  $b = f_*(a_{t_0}, \dots, a_{t_{n-1}})$  where  $t_0, \dots, t_{n-1} \in I_2$  and  $f_*$  a definable function, without loss of generality with  $n$  minimal hence  $\{a_{t_0}, \dots, a_{t_{n-1}}\}$  is transcendently independent and with no repetitions and  $b$  is not algebraic over  $\{a_{t_0}, \dots, a_{t_{n-1}}\} \setminus \{a_{t_\ell}\}$  for  $\ell < n$ . So  $\{t_0, \dots, t_{n-1}\} \not\subseteq I_u^2$  and let  $\ell < n$  be such that  $t_\ell \notin I_u^2$  hence there are  $s_0 <_{I_2} s_1$  such that  $t_\ell \in (s_0, s_1)_{I_2}$

and  $(s_0, s_1)_{I_2} \cap I_u^2 = \emptyset$  (recall  $\boxtimes_4(a), (b)$  and  $(*)_6$  about cofinality  $\aleph_0$  and  $I_2$  being dense). Also without loss of generality  $\{t_0, \dots, t_{n-1}\} \cap (s_0, s_1)_{I_2} = \{t_\ell\}$ , now the function  $c \mapsto f_*^M(a_{t_0}, \dots, a_{t_{\ell-1}}, c, a_{t_{\ell+1}}, \dots, a_{t_{n-1}})$  for  $c \in (a_{s_0}, a_{s_1})_I$  is increasing or decreasing (cannot be constant by the minimality on  $n$  and the elimination of quantifiers for real closed fields and the transcendental independence of  $\{t_0, \dots, t_{n-1}\}$ ). So we can find  $s'_0, s'_1$  such that  $s_0 <_{I_2} s'_0 <_{I_2} t_\ell <_{I_2} s'_1 <_{I_2} s_1$  such that  $X := \{f_*^M(a_{t_0}, \dots, a_{t_{\ell-1}}, c, a_{t_{\ell+1}}, \dots, a_{t_{n-1}}) : c \in (a_{s'_0}, a_{s'_1})_I\}$  is included in  $I_{u,x}$ . Again as the function defined above is monotonic on  $(a_{s'_0}, a_{s'_1})_I$  so for some value  $b' \in (a_{s'_0}, a_{s'_1})$  we have  $b <_I b'$ . But  $b$  is last in  $I_{u,x}$  by our assumption toward contradiction hence  $(b, b')_{I_u} \cap I_u = \emptyset$ . But this is impossible as all members of  $\{f(a_{t_0}, \dots, a_{t_{\ell-1}}, c, a_{t_{\ell+1}}, \dots, a_{t_{n-1}}) : c \in (a_{s'_1}, a_{s'_2})_I\}$  realize the same cut of  $I_u$  so  $(*)_8$  holds.]

$(*)_9$  if  $u$  is  $\mu$ -reasonable,  $x \in I \setminus I_u$  then also the inverse of  $I_{u,x}$  has cofinality  $\aleph_0$ .

[Why? Similarly to the proof of  $(*)_7 + (*)_8$  or note that the mapping  $y \mapsto -y$  (defined in  $M$ ) maps  $I_u$  onto itself and is an isomorphism from  $I$  onto its inverse.]

$(*)_{10}$  if  $u$  is  $\mu$ -reasonable, then  $I_u$  is unbounded in  $I$  from below and from above.

[Why? Easy.]

$(*)_{11}$  if  $h, u_1, u_2$  are as in clauses (a),(b),(c) below then the function  $h_4$  defined below is (well defined and) is, recalling  $(*)_2(g)$ , an order preserving function from  $\hat{I}_{u_1}$  onto  $\hat{I}_{u_2}$  mapping  $u_1$  onto  $u_2$  and also the functions  $h_0, h_1, h_2^*, h_2, h_3$  are as stated where

- (a)  $u_1, u_2 \subseteq \zeta$  are  $\mu$ -reasonable
- (b)  $h$  is an order preserving function from  $u_1$  onto  $u_2$
- (c)  $(\alpha)$  for  $\alpha \in u_1$ , we have  $\text{cf}(\alpha) \geq \theta_1 \Leftrightarrow \text{cf}(h(\alpha)) \geq \theta_1$   
 $(\beta)$  if  $\gamma \in u_1$  then  $(\forall \alpha < \gamma)(\exists \beta \in u_1)(\alpha \leq \beta < \gamma)$  iff  $(\forall \alpha < h(\gamma))(\exists \beta \in u_2)(\alpha \leq \beta < h(\gamma))$
- (d)  $(\alpha)$   $h_1$  is the induced order preserving function from  $I_{u_1}^1$  onto  $I_{u_2}^1$ , i.e.,  $h_1((\ell, \beta')) = (\ell, \beta'')$  when  $h(\beta') = \beta'' < \zeta$  or  $\beta' = \beta'' \in [\zeta, \zeta + \omega)$ ;  
 $(\beta)$  let  $h_0$  be the partial function from  $\zeta + \omega$  into  $\zeta + \omega$  such that  $h_0(\alpha) = \beta \Leftrightarrow (\exists \ell)[h_1((\ell, \alpha)) = (\ell, \beta)]$
- (e)  $h_2^*$  is the order preserving function from  $I_{u_1}^{*,2}$  onto  $I_{u_2}^{*,2}$  defined by: for  $\eta \in I_{u_1}^{*,2}$ ,  $h_2^*(\eta) = \langle h_1(\eta(\ell)) : \ell < \ell g(\eta) \rangle = \langle (\ell^{\eta(\ell)}, h_0(\alpha^{\eta(\ell)})) : \ell < \ell g(\eta) \rangle$ , recalling (d)
- (f)  $h_2 = h_2^* \upharpoonright I_{u_1}^2$  is an order preserving function from  $I_{u_1}^2$  onto  $I_{u_2}^2$

- (g)  $h_3$  is the unique isomorphism from the real closed field  $M_{I_{u_1}^2}$  onto the real closed field  $M_{I_{u_2}^2}$  mapping  $a_t$  to  $a_{h_2(t)}$  for  $t \in I_{u_1}^2$ , where for  $I' \subseteq I_2$  we let  $M_{I'} \subseteq M$  be the real closure of  $\{a_t : t \in I'\}$  inside  $M$
- (h)  $h_4$  is the map defined by:  
 $h_4(x) = y$  iff  
 ( $\alpha$ )  $x \in I_{u_1} \wedge y = h_3(x)$  or  
 ( $\beta$ ) for some  $a \in I \setminus I_{u_1}, b \in I \setminus I_{u_2}$  we have  $x = I_{u,a}, y \in I_{u,b}$  and  $(\forall c \in I_u)(c <_I a \equiv h_3(c) <_I b)$
- (i)  $\hat{I}_{u_1} = \text{Dom}(h_4)$  and  $\hat{I}_{u_2} = \text{Rang}(h_4)$  ordered naturally.

[Why? Trivially  $h_1$  is an order preserving function from  $I_{u_1}^1$  onto  $I_{u_2}^1$ . Recall  $I_{u_\ell}^{2,*} = \{\eta \in I_2^* : \eta(\ell) \in I_{u_\ell}^1 \text{ for } \ell < \ell g(\eta)\}$ . So obviously  $h_2^*$  is an order preserving function from  $I_{u_1}^{*,2}$  onto  $I_{u_2}^{*,2}$ . Now  $h_2 = h_2^* \upharpoonright I_{u_1}^2$ , but does it map  $I_{u_1}^2$  onto  $I_{u_2}^2$ ? we have excluded some members of  $I_{u_2}^{*,2}$  by  $\otimes$  above. But by clauses (c) and (d)( $\alpha$ ) of the assumption being excluded/not excluded is preserved by the natural mapping, i.e.,  $h_2^*$  maps  $I_{u_1}^2$  onto  $I_{u_2}^2$  hence  $h_2 = h_2^* \upharpoonright I_{u_1}^2$  is an isomorphism from  $I_{u_1}^1$  onto  $I_{u_2}^1$ . Also by  $(*)_1$  being the real closure of the ordered field  $M_0$ , and the uniqueness of “the real closure”  $h_3$  is the unique isomorphism from the real closed field  $M_{I_{u_1}^2}$  onto  $M_{I_{u_2}^2}$  mapping  $a_t$  to  $a_{h_2(t)}$  for  $t \in I_{u_1}^2$ .

Let  $\langle (\mathcal{U}_\varepsilon^1, \mathcal{U}_\varepsilon^2) : \varepsilon < \varepsilon^* \rangle$  list the pairs  $(\mathcal{U}_1, \mathcal{U}_2)$  such that:

- $\otimes_{10}$  (a)  $\mathcal{U}_\ell$  has the form  $I_{u_\ell, x}$  for some  $x \in I \setminus I_{u_\ell}$  for  $\ell = 1, 2$   
 (b) for every  $a \in I_{u_1}, (\exists y \in \mathcal{U}_1)(a <_I y) \Leftrightarrow (\exists y \in \mathcal{U}_2)(h_2(a) <_I y)$ .

Now

- $\otimes_{11}$   $\langle \mathcal{U}_\varepsilon^\ell : \varepsilon < \varepsilon^* \rangle$  is a partition of  $I \setminus I_{u_\ell}$  for  $\ell = 1, 2$ .

[Why? First, note the parallel claim for  $I_1$ . For this note that  $h_1((\ell, 0)) = (\ell, 0)$  as  $0 \in u_1 \cap u_2$  as  $u_1, u_2$  are  $\mu$ -reasonable, see clause (e) of  $(*)_4$  and  $h_1((\ell, \alpha)) = (\ell, \beta) \Leftrightarrow h_1((\ell, \alpha + 1)) = (\ell, \beta + 1)$ , by clause (b) of  $(*)_4$  and if  $h((\ell, \delta_1)) = (\ell, \delta_2), \delta_1$  is a limit (equivalently  $\delta_2$  is limit) then

$$\delta_1 = \sup\{\alpha < \delta : (\ell, \alpha) \in I_{u_1}^1\} \Leftrightarrow \delta_2 = \sup\{\alpha < \delta : (\ell, \alpha) \in I_{u_2}^1\}.$$

Second, note the parallel claim for  $h_2, I_{u_\ell}^{*,2}, h_2^*$ .

Third, note the parallel claim for  $I_{u_\ell}^2, h_2$ .

Fourth, note the parallel claim for  $I_{u_\ell}, h_3$  (which is the required one).]

So it follows that

$\otimes_{12}$   $h_4$  is as promised.

So we are done proving  $(*)_{11}$ .

[Why? By clauses (b),(c) of  $(*)_{11}$ .]

$(*)_{12}$  if  $u_1, u_2$  are  $\mu$ -reasonable,  $h$  is an order preserving mapping from  $\hat{I}_{u_1}$  onto  $\hat{I}_{u_2}$  which maps  $I_{u_1}$  onto  $I_{u_2}$  then there is an automorphism  $h^+$  of the linear order  $I$  extending  $h \upharpoonright I_{u_1}$ .

[Why? Let  $\langle \mathcal{U}_\varepsilon^1 : \varepsilon < \varepsilon^* \rangle$  list  $\hat{I}_{u_1} \setminus I_{u_1}$  and  $\mathcal{U}_\varepsilon^2 = h(\mathcal{U}_\varepsilon^1)$ . Now for every  $\varepsilon$  we choose  $\langle a_{\varepsilon,n}^\ell : n \in \mathbb{Z} \rangle$  such that

- $\otimes_{13}$  (a)  $a_{\varepsilon,n}^\ell \in \mathcal{U}_\varepsilon^\ell$
- (b)  $a_{\varepsilon,n}^\ell <_I a_{\varepsilon,n+1}^\ell$  for  $n \in \mathbb{Z}$
- (c)  $\{a_{\varepsilon,n}^\ell : n \in \mathbb{Z}, n \geq 0\}$  is unbounded from above in  $\mathcal{U}_\varepsilon^\ell$
- (d)  $\{a_{\varepsilon,n}^\ell : n \in \mathbb{Z}, n < 0\}$  is unbounded from below in  $\mathcal{U}_\varepsilon^\ell$ .

This is justified by  $u_\ell$  being  $\mu$ -reasonable by  $(*)_6$ ,  $\boxtimes_4$ . Now define  $h_5 : I \rightarrow I$  by:

$$\begin{aligned} h_5(x) &= h_4(x) \text{ if } x \in I_{u_1} \text{ and otherwise} \\ h_5(x) &= a_{\varepsilon,n}^2 + (a_{\varepsilon,n+1}^2 - a_{\varepsilon,n}^2)(x - a_{\varepsilon,n}^1)/(a_{\varepsilon,n+1}^1 - a_{\varepsilon,n}^1) \end{aligned}$$

if  $a_{\varepsilon,n}^1 \leq_{I_2} x < a_{\varepsilon,n+1}^1$  and  $n \in \mathbb{Z}$ .

Now check using linear algebra.]

$(*)_{13}$   $(^\mu I)/\mathcal{C}_{I,\mu}^{\text{aut}}$  has  $\leq 2^\mu$  members recalling that  $f_1 \mathcal{C}_{I,h}^{\text{aut}} f_2$  iff  $f_1, f_2$  are functions from  $\mu$  into  $I$  and for some automorphism  $h$  of  $I$  we have  $(\forall \alpha < \mu)(h(f_1(\alpha)) = f_2(\alpha))$

[Why? Should be clear recalling  $|I_u^1| \leq \mu$ , recalling  $(*)_5, (*)_{11}, (*)_{12}$ .]

So we have finished proving part (1) of 5.1.

2) Really the proof is included in the proof of part (1). That is, given  $I' \subseteq I$  of cardinality  $< \theta_2$  by  $(*)_5$  there is a  $\mu$ -reasonable  $u \subseteq \zeta$  such that  $I' \subseteq I_u$  and  $|u| = \mu + |I'|$ . Now clearly

$(*)_{14}$  for  $\mu$ -reasonable  $u \subseteq \zeta$ , the family  $\{I_{u,x}^2 : x \in I_2 \setminus I_u^2\}$  has  $\leq \mu + |u|$  members.

[Why? By  $\boxtimes_5$ .]

$(*)_{15}$  for a  $\mu$ -reasonable  $u \subseteq \zeta$ , the family  $\{I_{u,x} : x \in I \setminus I_u\}$  has  $\leq \mu$  members.

[Why? By  $(*)_{16}$  below.]

- $(*)_{16}$  if  $u$  is  $\mu$ -reasonable then  $I_{u,b_1} = I_{u,b_2}$  when
- (a)  $b_k = f(a_{t_{k,0}}, \dots, a_{t_{k,n-1}})$  for  $k = 1, 2$
  - (b)  $f$  a definable function in  $M$
  - (c)  $t_{k,0} <_{I_2} \dots <_{I_2} t_{k,n-1}$  for  $k = 1, 2$
  - (d)  $t_{1,\ell} \in I_u^2 \vee t_{2,\ell} \in I_u^2 \Rightarrow t_{1,\ell} = t_{2,\ell}$
  - (e) if  $t_{1,\ell} \notin I_u^2$  then  $I_{u,t_{1,\ell}}^2 = I_{u,t_{2,\ell}}^2$  for  $\ell = 0, \dots, n-1$ .

[Why? Use the proof of  $(*)_{11}$ , for  $u_1 = u = u_2, h = \text{id}_{u_2}$  so  $\mathcal{U}_\varepsilon^1 = \mathcal{U}_\varepsilon^2$  for  $\varepsilon < \varepsilon^*$ .

By the assumptions for each  $\ell$  there is  $\varepsilon$  such that  $a_{t_{\varepsilon,1,\ell}}, a_{t_{2,\ell}} \in \mathcal{U}_\varepsilon^1 = \mathcal{U}_\varepsilon^2$ . Now for each  $\varepsilon < \varepsilon^*$  there is an automorphism  $\pi_\varepsilon$  of  $\mathcal{U}_\varepsilon^1$  as a linear order mapping  $t_{1,\ell}$  to  $t_{2,\ell}$  if  $t_{1,\ell} \in \mathcal{U}_\varepsilon^1$ . Let  $\pi = \cup \{\pi_\varepsilon : \varepsilon < \varepsilon^*\} \cup \text{id}_{I_u}$ .]

- $(*)_{17}$  if  $n < \omega, t_0^\ell <_I t_1^\ell <_I \dots <_I t_{n-1}^\ell$  for  $\ell = 1, 2, I_{u,t_k^1} = I_{u,t_k^2}$  for  $k = 0, 1, \dots, n-1$  then for some automorphism  $g$  of  $I$  over  $I_u$  we have  $k < n \Rightarrow g(t_k^1) = t_k^2$ .

[Why? We shall use  $g$  such that  $g \upharpoonright I_u = \text{id}_{I_u}$  and  $g \upharpoonright I_{u,x}$  is an automorphism of  $I_{u,x}$  for each  $x \in I \setminus I_u$ . Clearly it suffices to deal with the case  $\{t_k^\ell : \ell < n \text{ and } \ell \in \{1, 2\}\} \subseteq I_{u,x}$  for one  $x \in I \setminus I_u$ . We choose  $s_1 < s_2$  from  $I_{u,x}$  such that  $s_1 <_I t_k^\ell < s_2$  for  $\ell = 1, 2$ . We choose  $g \upharpoonright I_{u,x}$  such that it is the identity on  $\{s \in I_{u,x} : s \leq_I s_1 \text{ or } s_2 \leq_I s\}$ , now stipulates  $t_{-1} = s_1, t_n = s_2$  and maps  $(t_k^1, t_{k+1}^1)_I$  onto  $(t_k^2, t_{k+1}^2)_I$  for  $k = -1, 0, \dots, n-1$  as in the definition above.]

So we have completed the proof of part (2) of 5.1.

3) Obvious from the Definition (0.14(9)) and the construction.

4) First

- $\odot_1$  there is  $J_1^* \subseteq I$  of cardinality  $\mu^+$  such that: for every  $J_2^* \subseteq I$  of cardinality  $\leq \mu$  there is an automorphism  $\pi$  of  $I$  which maps  $J_2^*$  into  $J_1^*$ .

[Why? Let  $u = \mu^+ \times \mu^+ \subseteq \zeta$  and let  $J_1^* = I_u$ . Clearly  $u$  has cardinality  $\mu^+$  and so does  $J_1^* = I_u$ . So suppose  $J_2^* \subseteq I$  has cardinality  $\leq \mu$ . There is  $u_2 \subseteq \zeta$  of cardinality  $\mu$  such that  $J_2^* \subseteq I_{u_2}$  and without loss of generality  $u_2$  is reasonable. We define an increasing function  $h$  from  $u_2$  into  $u_1$ , by defining  $h(\alpha)$  by induction on  $\alpha$ :

- $(*)_{17}$  if  $\text{cf}(\alpha) \leq \mu$  then  $h(\alpha) = \cup \{h(\beta) + 1 : \beta \in u_2 \cap \alpha\}$
- $(*)_{18}$  if  $\text{cf}(\alpha) > \mu$  then  $h(\alpha) = \cup \{h(\beta) + 1 : \beta \in u_2 \cap \alpha\} + \mu^+$ .

Let  $u_1 := \{h(\alpha) : \alpha \in u_2\}$  so  $u_1 \subseteq u$ . Now  $h, u_1, u_2$  satisfies clauses (a),(b),(c) of  $(*)_{11}$  hence  $h_1, h_2^*, h_2, h_3, h_4, \hat{I}_{u_1}, \hat{I}_{u_2}$  are as there.

By  $(*)_{12}$  there is an isomorphism  $h^+$  of  $I$  which extends  $h_4$ ; now does  $h^+$  map  $J_2^*$  into  $J_1^*$ ? Yes, as  $J_2^* \subseteq I_{u_2}$  and  $h^+ \upharpoonright I_{u_2}$  is an isomorphism from  $I_{u_2}$  onto  $I_{u_1}$  but  $I_{u_1} \subseteq I_u, I_u = J_1^*$ , so we are done proving  $\odot_1$ .]

Finally

$\odot_2$  part (4) of 5.1 holds, i.e. if  $I_0^* \subseteq I, |I_0^*| < \theta_2$  then for some  $I_1^* \subseteq I$  of cardinality  $\leq \mu^+ + |I_0^*|$  we have: for every  $J \subseteq I$  of cardinality  $\leq \mu$  there is an automorphism of  $I$  over  $I_0^*$  mapping  $J$  into  $I_1^*$ .

Why? Given  $I_0^* \subseteq I$  of cardinality  $< \theta_2$  we can find  $u_1 \subseteq \zeta$  of cardinality  $\mu + |I_0^*|$  such that  $I_0^* \subseteq I_{u_1}$ . By  $(*)_5$  we can find a  $\mu$ -reasonable set  $u_2 \subseteq \zeta$  of cardinality  $\mu + |u_1|$  such that  $u_1 \subseteq u_2$ .

Let  $\langle \mathcal{U}_\varepsilon : \varepsilon < \varepsilon^* \rangle$  list the sets of the form  $I_{u_2, x}, x \in I_2 \setminus I_{u_1}$ , so by  $(\square_5)$   $\varepsilon^* \leq \mu + |I_0^*|$ . For each  $\varepsilon$  we choose  $\langle a_{\varepsilon, n} : n \in \mathbb{Z} \rangle$  as in  $(*)_{13}$  from the proof of  $(*)_{12}$ . For each  $\varepsilon < \varepsilon^*$  and  $n \in \mathbb{Z}$  let  $\pi_{\varepsilon, n}$  be an isomorphism from  $I$  onto  $(a_{\varepsilon, n}, a_{\varepsilon, n+1})I$ , exists by the properties of ordered fields. Let  $J_1^* \subseteq I$  be as in  $\odot_1$  above and let  $I_2^* = I_1^* \cup \{a_{\varepsilon, n} : \varepsilon < \varepsilon^* \text{ and } n < \omega\} \cup \{\pi_{\varepsilon, n}(J_1^*) : \varepsilon < \varepsilon^* \text{ and } n \in \mathbb{Z}\}$ . Easily,  $I_2^*$  is as required.

5) By 0.12.

$\square_{5.1}$

*Remark.* Concerning  $(*)_{11}$ , we could have used more time

$(*)'_{11}$   $h_2$  is an order preserving function from  $I_{u_1}^2$  onto  $I_{u_2}^2$  and  $h_3$  is an isomorphism from  $I_{u_1}$  onto  $I_{u_2}$  and  $h_1$  is an order preserving mapping from  $\hat{I}_{u_2}$  onto  $\hat{I}_{u_2}$ .

## §6 LINEAR ORDERS AND EQUIVALENCE RELATIONS

This section deals with a relative of the stability spectrum. We ask: what can be the number of equivalence classes in  ${}^\mu I$  for an equivalence relation on  ${}^\mu I$  which is so called “invariant”, in fact definable (essentially by a quantifier free infinitary formula, mainly for well ordered  $I$ ).

It is done in a very restricted context, but via EM-models has useful conclusions, for a.e.c. and also for a.e.c. with amalgamation; i.e. it is used in 7.8.

There are two versions; one for well ordering and one for the class of linear orders both expanded by unary relations.

On  $\tau_{\alpha(*)}^*, K_{\tau_{\alpha(*)}^*}^{\text{lin}}$  see 0.14(4). We may replace sequences, i.e.  $\text{inc}_J(I)$  by subsets of  $I$

of cardinality  $|J|$ , this may help to eliminate  $2^{|J|}$  later, but at present it seems not to help in the final bounds in §7. We do here only enough for §7.

**6.1 Context.** We fix  $\alpha(*), \bar{u}^* = (u^-, u^+)$  such that

- (a)  $\alpha(*)$  is an ordinal  $\geq 1$
- (b)  $u^- \subseteq \alpha(*)$
- (c)  $u^+ \subseteq \alpha(*)$ .

**6.2 Remark.** 1) The main cases are

- (A)  $\alpha(*) = 1$ , so  $K_{\tau_{\alpha(*)}}^{\text{lin}}$  is the class of linear orders
- (B)  $\alpha(*) = 2, u^+ = \emptyset, u^- = \{0\}$ .

2) Usually the choice of the parameters does not matter.

**6.3 Definition.** 1) For  $I, J \in K_{\tau_{\alpha(*)}}^{\text{lin}}$ , i.e. both linear orders expanded by a partition  $P_\alpha(\alpha < \alpha(*))$ , pedantically the interpretation of the  $P_\alpha$ 's, let  $\text{inc}'_J(I)$  be the set of embedding of  $J$  into  $I$ ; see below, we denote members by  $h$ .

2) Recalling  $\bar{u}^* = (u^-, u^+)$  where  $u^- \cup u^+ \subseteq \alpha(*)$  let  $\text{inc}_{\bar{u}^*}^J(I)$  be the set of  $h$  such that

- (a)  $h$  is an embedding of  $J$  into  $I$ , i.e. one-to-one, order preserving function mapping  $P_\alpha^J$  into  $P_\alpha^I$  for  $\alpha < \alpha(*)$
- (b) if  $\alpha \in u^-$  and  $t \in P_\alpha^J$  and  $s <_I h(t)$  then for some  $t_1 <_J t$  we have  $s \leq_I h(t_1)$
- (c) if  $\alpha \in u^+$  and  $t \in P_\alpha^J$  and  $h(t) <_I s$  then for some  $t_1$  we have  $t <_J t_1$  and  $h(t_1) \leq_I s$ .

Concerning  $\bar{u}^*$

**6.4 Observation.** 1) For any  $h \in \text{inc}_{\bar{u}^*}^J(I)$

- (a) if  $t$  is the successor of  $s$  in  $J$  (i.e.  $s <_J t$  and  $(s, t)_J = \emptyset$ ) and  $t \in P_\alpha^J, \alpha \in u^-$  then  $h(t)$  is the successor of  $h(s)$  in  $I$
- (b) if  $\langle t_i : i < \delta \rangle$  is  $<_J$ -increasing with limit  $t_\delta \in J$  (i.e.  $i < \delta \Rightarrow t_i <_J t_\delta$  and  $\emptyset = \cap \{(t_i, t_\delta)_J : i < \delta\}$ ) and  $t_\delta \in P_\alpha^J, \alpha \in u^-$  then  $\langle h(t_i) : i < \delta \rangle$  is  $<_I$ -increasing with limit  $h(t_\delta)$  in  $I$
- (c) if  $t$  is the first member of  $J$  and  $t \in P_\alpha^J, \alpha \in u^-$  then  $h(t)$  is the first member of  $I$ .



2) If  $h_1, h_2 \in \text{inc}_J^{\bar{u}^*}(I)$  then

- (a) if  $t$  is the successor of  $s$  in  $J$  and  $t \in P_\alpha^J, \alpha \in u^-$  then  $h_1(s) = h_2(s) \Leftrightarrow h_1(t) = h_2(t)$  and  $h_1(s) <_I h_2(s) \Leftrightarrow h_1(t) <_I h_2(t)$  and  $h_1(s) >_I h_2(s) \Leftrightarrow h_1(t) >_I h_2(t)$
- (b) if  $\langle t_i : i < \delta \rangle$  is  $<_J$ -increasing with limit  $t_\delta$  and  $t_\delta \in P_\alpha^J, \alpha \in u^-$ , then  $(\forall i < \delta)(h_1(t_i) = h_2(t_i)) \Rightarrow h_1(t_\delta) = h_2(t_\delta)$  moreover  $(\forall i < \delta)(\exists j < \delta)(h_1(t_i) <_I h_2(t_j) \wedge h_2(t_i) <_I h_1(t_j)) \Rightarrow h_1(t_\delta) = h_2(t_\delta)$  and also  $(\exists j < \delta)(\forall i < \delta)(h_1(t_i) <_I h_2(t_j)) \Rightarrow h_1(t_\delta) <_I h_2(t_\delta)$ .

3) Similar to parts (1) + (2) for  $\alpha \in u^+$  (inverting the orders of course).

4)  $\text{inc}'_I(J) = \text{inc}_I^{(\emptyset, \emptyset)}(J)$ .

*Proof.* Straight (and see the proof of 6.7). □<sub>6.4</sub>

**6.5 Convention.** 1)  $\alpha(*), \bar{u}^*$  will be constant so usually we shall not mention them, e.g. write  $\text{inc}_J(I)$  for  $\text{inc}_I^{\bar{u}^*}(I)$  and pedantically below we should have written  $\mathbf{e}^{\bar{u}^*}(J, I), \mathbf{e}_*^{\bar{u}^*}(J)$  and also in notions like reasonable and wide in Definition 6.10 mention  $\bar{u}^*$ .

2)  $I, J$  denote members of  $K_{\tau_{\alpha(*)}^*}^{\text{lin}}$ .

Below we use mainly “ $e$ -pairs” (and weak  $e$ -pairs and the reasonable case).

**6.6 Definition.** 1) let  $\mathbf{e}(J)$  be the set of equivalence relations on some subset of  $J$  such that each equivalence class is a convex subset of  $J$ .

2) For  $h_1, h_2 \in \text{inc}_J(I)$  we say that  $(h_1, h_2)$  is a strict  $e$ -pair (for  $(I, J)$ ) when  $e \in \mathbf{e}(J)$  and  $(h_1, h_2)$  satisfies

- (a)  $s \in J \setminus \text{Dom}(e)$  iff  $h_1(s) = h_2(s)$
- (b) if  $s <_J t$  and  $s/e \neq t/e$  (so  $s, t \in \text{Dom}(e)$ ) then  $h_1(s) <_I h_2(t)$  and  $h_2(s) <_I h_1(t)$
- (c) if  $s <_J t$  and  $s/e = t/e$  (so  $s, t \in \text{Dom}(e)$ ) then  $h_1(t) <_I h_2(s)$ .

2A) We say that  $(h_1, h_2)$  is a strict  $(e, \mathcal{Y})$ -pair where  $e \in \mathbf{e}(J)$  and  $\mathcal{Y} \subseteq \text{Dom}(e)/e$  when clauses (a)+(b) from part (2) hold and

- (c)' if  $s <_J t$  and  $s/e = t/e$  (so  $s, t \in \text{Dom}(e)$ ) then  $(h_1(t) <_I h_2(s)) \equiv (s/e \in \mathcal{Y}) \equiv (h_1(s) < h_2(t))$ .

2B) We say that  $(h_1, h_2)$  is an  $e$ -pair when  $(h_1, h_2)$  is a strict  $(e, \mathcal{Y})$ -pair for some  $\mathcal{Y}$  (this relation is symmetric, see below).

3) We say that  $(h_1, h_2)$  is a weak  $e$ -pair where  $h_1, h_2 \in \text{inc}_J(I)$  when clauses (a),(b) hold (this, too, is symmetric!)

4) For  $h_1, h_2 \in \text{inc}_J(I)$ , let  $e = \mathbf{e}(h_1, h_2)$  be the (unique)  $e \in \mathbf{e}(J)$  such that (see 6.8(1) below)

- (a)  $\text{Dom}(e) = \{s \in J : h_1(s) \neq h_2(s)\}$
- (b)  $(h_1, h_2)$  is a weak  $e$ -pair
- (c) if  $e' \in \mathbf{e}(J)$  and  $(h_1, h_2)$  is a weak  $e'$ -pair then  $\text{Dom}(e) \subseteq \text{Dom}(e')$  and  $e$  refines  $e' \upharpoonright \text{Dom}(e)$ .

5) If  $e \in \mathbf{e}(J)$  and  $\mathcal{Y} \subseteq \text{Dom}(e)/e$  then we let  $\text{set}(\mathcal{Y}) = \{s \in J : s/e \in \mathcal{Y}\}$  and  $e \upharpoonright \mathcal{Y} = e \upharpoonright \text{set}(\mathcal{Y})$ .

6) Let  $\mathbf{e}(J, I)$  be the set of  $e \in \mathbf{e}(J)$  such that there is an  $e$ -pair.

7) Let  $\mathbf{e}_*(J) = \cup \{\mathbf{e}(J, I) : I \in K_{\tau_{\alpha(*)}}^{\text{lin}}\}$ .

Concerning  $\bar{u}^*$

6.7 *Observation.* Assume that  $e \in \mathbf{e}(J, I)$ .

0)

- (a) If  $t$  is the first member of  $J$  and  $t \in P_\alpha^J, \alpha \in u^-$  then  $t \notin \text{Dom}(e)$ .
- (b) If  $t \in \text{Dom}(e)$  and  $t$  is the first member of  $t/e$  and  $t \in P_\alpha^J$  then  $\alpha \notin u^-$ .

1) If  $t$  is the  $<_J$ -successor of  $s$  and  $t \in P_\alpha^J, \alpha \in u^-$  then  $s \in \text{Dom}(e) \Leftrightarrow t \in \text{Dom}(e)$  and  $s \in \text{Dom}(e) \Rightarrow s \in t/e$ .

2) If  $\langle t_i : i < \delta \rangle$  is  $<_J$ -increasing with limit  $t_\delta$  and  $t_\delta \in P_\alpha^J$  and  $\alpha \in u^-$  then:

- (a) if  $(\forall i < \delta)(t_i \notin \text{Dom}(e))$  then  $t_\delta \notin \text{Dom}(e)$
- (b) if  $(\forall i < \delta)(\neg t_i e t_{i+1})$  or just  $(\forall i < \delta)(\exists j < \delta)(i < j \wedge \neg t_i e t_j)$  then  $t_\delta \notin \text{Dom}(e)$
- (c) if  $(\forall i < \delta)(t_i \in t_0/e)$  then  $t_\delta \in t_0/e$ .

3) Similar to parts (0),(1),(2) when  $\alpha \in u^+$  (inverting the order, of course).

4)  $\mathbf{e}_*(J)$  is the family of  $e \in \mathbf{e}(J)$  satisfying the requirements in parts (0),(1),(2),(3) above so if  $\bar{u}^* = (\emptyset, \emptyset)$  then  $\mathbf{e}_*(J) = \mathbf{e}(J)$ .

*Proof.* Easy by 6.4, e.g.

Part (1): We are assuming  $e \in \mathbf{e}(J, I)$  hence by Definition 6.6 there is an  $e$ -pair  $(h_1, h_2)$  where  $h_1, h_2 \in \text{inc}_J(I)$ . Now for  $\ell = 1, 2$ , clearly  $h_\ell(s), h_\ell(t) \in I$  and as

$s <_J t$  we have  $h_\ell(s) < h_\ell(t)$ . Now if  $h_\ell(t)$  is not the  $<_I$ -successor of  $h_\ell(s)$  then there is  $s'_\ell \in (h_\ell(s), h_\ell(t))_I$  hence by clause (b) of Definition 6.3(2) there is  $s_\ell^* \in [s, t]_J$  such that  $s'_\ell \leq_I h_\ell(s_\ell^*) <_I h_\ell(t)$  so as  $h_\ell(s) <_I s'_\ell$  we have  $h_\ell(s) <_I h_\ell(s_\ell^*) <_I h_\ell(t)$  hence  $s <_I s_\ell^* <_J t$ , contradiction to the assumption “ $t$  is the successor of  $s$  in  $J$ ”. So indeed  $h_\ell(t)$  is the successor of  $h_\ell(s)$  in  $I$ .

As this holds for  $\ell = 1, 2$ , clearly  $h_1(s) = h_2(s) \Leftrightarrow h_1(t) = h_2(t)$  but by Definition 6.3(2) we know  $s \in \text{Dom}(e) \Leftrightarrow (h_1(s) \neq h_2(s))$  and similarly for  $t$  hence  $s \in \text{Dom}(e) \Leftrightarrow t \in \text{Dom}(e)$ . Lastly, assume  $s, t \in \text{Dom}(e)$ , but  $s, t$  are not  $e$ -equivalent so by Definition 6.6(2) clause (b) we have  $h_1(s) <_I h_2(t) \wedge h_2(s) <_I h_1(t)$  clear contradiction.

Part 2: We leave clauses (a),(b) to the reader.

For clause (c) of part (2), if  $t_\delta \notin t_0/e$  then choose  $h_1, h_2 \in \text{inc}_J^{\bar{u}^*}(I)$  such that  $(h_1, h_2)$  is an  $e$ -pair, hence an  $(e, \mathcal{Y})$ -pair for some  $\mathcal{Y} \subseteq \text{Dom}(e)/e$ . If  $(t_0/e) \in \mathcal{Y}$  then  $h_2(t_0)$  is above  $\{h_1(t_i) : i < \delta\}$  by  $<_I$  so we have  $h_1(t_\delta) \leq_I h_2(t_0)$  but if  $t_\delta \notin t_0/e$  this contradicts clause (b) in Definition 6.6(2),(2A). The proof when  $t_0/e \notin \mathcal{Y}$  is similar.  $\square_{6.7}$

*6.8 Observation.* Let  $h_1, h_2 \in \text{inc}_J(I)$  and  $e \in \mathbf{e}(J)$ .

- 1)  $\mathbf{e}(h_1, h_2)$  is well defined.
- 2)  $(h_1, h_2)$  is a strict  $(e, \mathcal{Y}_1)$ -pair iff  $(h_2, h_1)$  is a strict  $(e, \mathcal{Y}_2)$ -pair when  $(\mathcal{Y}_1, \mathcal{Y}_2)$  is a partition of  $\text{Dom}(e)/e$ .
- 3)  $(h_1, h_2)$  is a strict  $e$ -pair iff  $(h_2, h_1)$  is a strict  $(e, \emptyset)$ -pair.
- 4)  $(h_1, h_2)$  is an  $e$ -pair iff  $(h_2, h_1)$  is an  $e$ -pair.
- 5)  $(h_1, h_2)$  is a weak  $e$ -pair iff  $(h_2, h_1)$  is a weak  $e$ -pair.
- 6) If  $(h_1, h_2)$  is a strict  $e$ -pair then  $(h_1, h_2)$  is an  $e$ -pair which implies  $(h_1, h_2)$  being a weak  $e$ -pair.
- 7) If  $e_\alpha \in \mathbf{e}(J)$  for  $\alpha < \alpha^*$ , then  $e := \cap \{e_\alpha : \alpha < \alpha^*\} = \{(s, t) : s, t \text{ are } e_\alpha\text{-equivalent for every } \alpha < \alpha^*\}$  belongs to  $\mathbf{e}(J)$  with  $\text{Dom}(e) = \cap \{\text{Dom}(e_\alpha) : \alpha < \alpha^*\}$ .
- 8) If  $e \in \mathbf{e}(J, I)$  then for every  $\mathcal{Y} \subseteq \text{Dom}(e)/e$  also  $e \upharpoonright \text{set}(\mathcal{Y})$  belongs to  $\mathbf{e}(J, I)$  and there is a strict  $(e \upharpoonright \text{set}(\mathcal{Y}))$ -pair  $(h'_1, h'_2)$ ; moreover, for every  $\mathcal{Y}_1 \subseteq \mathcal{Y}$  there is a strict  $(e \upharpoonright \text{set}(\mathcal{Y}_1), \mathcal{Y}_1)$ -pair.  $\square_{6.8}$

*Proof.* Easy, e.g.:

- 1) Let

$e = \{(s_1, s_2) : h_1(s_\ell) \neq h_2(s_\ell) \text{ for } \ell = 1, 2 \text{ and if } s_1 \neq s_2 \text{ then}$   
     for some  $t_1 <_J t_2$  we have  $\{s_1, s_2\} = \{t_1, t_2\}$   
     and there is no initial segment  $J'$  of  $J$  such that  
 $J' \cap \{t_1, t_2\} = \{t_1\}$  and  
 $(\forall t' \in J')(\forall t'' \in J \setminus J')[h_1(t') <_I h_2(t'') \wedge h_2(t') <_I h_1(t'')]\}$ .

Clearly  $e$  is an equivalence relation on  $\{t \in J : h_1(t) \neq h_2(t)\}$  and each equivalence class is convex hence  $e_1 \in \mathbf{e}(J)$ , so clauses (a),(b) of 6.6(1),(4) holds. Easily  $e$  is as required.

8) Let  $(h_1, h_2)$  be an  $e$ -pair and  $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$  be a partition of  $\text{Dom}(e)/e$ . We define  $h'_1, h'_2 \in \text{inc}_J(I)$  as follows, for  $\ell \in \{1, 2\}$

- (a) if  $t \in J \setminus \text{Dom}(e)$  then  $h'_\ell(t) = h_1(t) (= h_2(t))$
- (b) if  $t \in \text{set}(\mathcal{Y}_1)$  then  $h'_\ell(t) = h_1(t)$
- (c) if  $t \in \text{set}(\mathcal{Y}_2)$  then  $h'_\ell(t)$  is  $\min\{h_1(t), h_2(t)\}$  if  $\ell = 1$ , and is  $\max\{h_1(t), h_2(t)\}$  if  $\ell = 2$
- (d) if  $t \in \text{set}(\mathcal{Y}_3)$  then  $h'_\ell(t)$  is  $\max\{h_1(t), h_2(t)\}$  if  $\ell = 1$  and is  $\min\{h_1(t), h_2(t)\}$  if  $\ell = 2$ .

Now  $(h'_1, h'_2)$  is a strict  $(e \upharpoonright (\text{set}(\mathcal{Y}_2) \cup \text{set}(\mathcal{Y}_3)), \mathcal{Y}_2)$ -pair, so we are done.  $\square_{6.8}$

**6.9 Definition.** 1) For a subset  $u$  of  $J \in K_{\alpha(*)}^{\text{lin}}$  we define  $e = e_{J,u} \in \mathbf{e}(J)$  on  $J \setminus u$  as follows:

$$s_1 e s_2 \text{ iff } (\forall t \in u)(t <_J s_1 \equiv t <_J s_2).$$

2) For  $I, J \in K_{\alpha(*)}^{\text{lin}}$ , we say that the pair  $(I, J)$  is non-trivial when:  $\mathbf{e}(J, I) \neq \emptyset$ .

**6.10 Definition.** 1) For  $h_0, \dots, h_{n-1} \in \text{inc}_J(I)$  let

$$\text{tp}_{\text{qf}}^J(\langle h_0, \dots, h_{n-1} \rangle, I) = \{(\ell, m, s, t) : s, t \in J \text{ and } h_\ell(s) < h_m(t)\}.$$

We may write  $\text{tp}_{\text{qf}}^J(h_0, \dots, h_{n-1}; I)$  and we usually omit  $J$  as it is clear from the context.

2) For  $h_1, h_2 \in \text{inc}_J(I)$  let  $\text{eq}(h_1, h_2) = \{s \in J : h_1(s) = h_2(s)\}$ .

3) We say that the pair  $(I, J)$  is a reasonable  $(\mu, \alpha(*))$ -base when:

- (a)  $I, J \in K_{\tau_{\alpha(*)}}^{\text{lin}}$ ,  $|J| \leq \mu$  and the pair  $(I, J)$  is non-trivial
- (b) if  $e \in \mathbf{e}(J, I)$  and  $h_1, h_2 \in \text{inc}_J(I)$  and  $(h_1, h_2)$  is an  $e$ -pair then we can find  $h'_1, h'_2, h'_3 \in \text{inc}_J(I)$  and  $\mathscr{Y} \subseteq \text{Dom}(e)/e$  such that
  - ( $\alpha$ )  $\text{tp}_{\text{qf}}((h'_1, h'_2), I) = \text{tp}_{\text{qf}}((h_1, h_2), I)$
  - ( $\beta$ )  $(h'_1, h'_3)$  and  $(h'_2, h'_3)$  are strict  $(e, \mathscr{Y})$ -pairs.

4) We say that the pair  $(I, J)$  is a wide  $(\lambda, \mu, \alpha(*))$ -base when:

- (a)  $I, J \in K_{\tau_{\alpha(*)}}^{\text{lin}}$ ,  $|J| \leq \mu$  and the pair  $(I, J)$  is non-trivial
- (b) for every  $e \in \mathbf{e}(J, I)$  there is a sequence  $\bar{h} = \langle h_\alpha : \alpha < \lambda \rangle$  such that
  - ( $\alpha$ )  $h_\alpha$  is an embedding of  $J$  into  $I$
  - ( $\beta$ ) if  $\alpha < \beta < \lambda$  then  $(h_\alpha, h_\beta)$  is an  $e$ -pair.

5) We say that the pair  $(I, J)$  is a strongly wide  $(\lambda, \mu, \alpha(*))$ -base when:

- (a)  $I, J \in K_{\tau_{\alpha(*)}}^{\text{lin}}$ , the pair  $(I, J)$  is non-trivial and  $J$  has cardinality  $\leq \mu$
- (b) for every  $e \in \mathbf{e}(J, I)$  and  $\mathscr{Y} \subseteq \text{Dom}(e)/e$  there is  $\bar{h} = \langle h_\alpha : \alpha < \lambda \rangle$  such that
  - ( $\alpha$ )  $h_\alpha \in \text{inc}_J(I)$
  - ( $\beta$ ) if  $\alpha < \beta$  then  $(h_\alpha, h_\beta)$  is a strict  $(e, \mathscr{Y})$ -pair.

6) Above we may omit  $\mu$  meaning  $\mu = |J|$  and we may omit  $\alpha(*)$ , as it is determined by  $J$  (and by  $I$ ), and then may omit “base” so in part (3) we say  $(I, J)$  is reasonable and in part (4) we say  $\lambda$ -wide and in part (5) say strongly  $\lambda$ -wide.

*6.11 Observation.* 1) If  $(I, J)$  is a reasonable  $(\mu, \alpha(*))$ -base then  $(I, J)$  is a reasonable  $(\mu', \alpha(*))$ -base for  $\mu' \geq \mu$ .

2) If  $(I, J)$  is a wide  $(\lambda, \mu, \alpha(*))$ -base and  $\lambda' \leq \lambda, \mu' \geq \mu$  then  $(I, J)$  is a wide  $(\lambda', \mu', \alpha(*))$ -base.

3) If  $(I, J)$  is a strongly wide  $(\lambda, \mu, \alpha(*))$ -base, then  $(I, J)$  is a wide  $(\lambda, \mu, \alpha(*))$ -base.

*Proof.* Obvious.

□<sub>6.11</sub>

**6.12 Claim.** 1) If  $\alpha(*) = 1$  and  $\mu \leq \zeta(*) < \mu^+ \leq \lambda$ , then the pair  $(\lambda \times \zeta(*), \zeta(*))$  is a reasonable  $(\mu, \alpha(*))$ -based which is a wide  $(\lambda, \mu, \alpha(*))$ -base.

2) If  $\alpha(*) = 2$  and  $\bar{u}^* = (\{0\}, \emptyset)$  as in 6.2 and  $\mu \leq \zeta(*) < \mu^+ < \lambda$  and  $\zeta'(*) = \zeta(*) \times 3$  and  $w \subseteq \zeta(*), w \neq \zeta(*)$  then the pair  $(I_{\mu, \lambda \times \zeta(*)}^{\text{lin}}, I_{\mu, \zeta(*), w}^{\text{lin}})$  is a reasonable  $(\mu, \alpha(*))$ -base which is a wide  $(\lambda, \mu, \alpha(*))$ -base where

(\*) for any ordinal  $\beta$  and  $w \subseteq \beta$  we define  $I = I_{\mu, \beta, w}^{\text{lin}}$ , a  $\tau_{\alpha(*)}^*$ -model (if  $w = \emptyset$  we may omit it)

( $\alpha$ ) its universe is  $\beta$

( $\beta$ ) the order is the usual one

( $\gamma$ )  $P_1^I = \{\alpha < \beta : \text{cf}(\alpha) > \mu \text{ or } \alpha \in w\}$ , (if we write  $I_{\geq \mu, \beta, w}^{\text{lin}}$  we mean here  $\text{cf}(\alpha) \geq \mu$ ).

*Proof.* 1) First:  $(I, J) = (\lambda \times \zeta(*), \zeta(*))$  is a wide  $(\lambda, \mu, \alpha(*))$ -base

Easily  $\mathbf{e}(J, I) \neq \emptyset, |J| \leq \mu$  and  $I, J \in K_{\tau_{\alpha(*)}^*}^{\text{lin}}$  so clause (a) of Definition 6.10(4) holds (recalling Definition 6.9(2)), so it suffices to deal with clause (b).

Let  $e \in \mathbf{e}(J, I)$  and define

$$u = \{\zeta < \zeta(*) : \zeta \in \text{Dom}(e) \text{ is minimal in } \zeta/e \\ \text{or } \zeta \in \zeta(*) \setminus \text{Dom}(e)\}.$$

Now for every  $\alpha < \lambda$  we define  $h_\alpha \in \text{inc}_J(I)$  as follows:

(a) if  $\zeta \in \zeta(*) \setminus \text{Dom}(e)$  then  $h_\alpha(\zeta) = \lambda \times \zeta$

(b) if  $\zeta \in \text{Dom}(e)$  and  $\varepsilon = \min(\zeta/e)$  then  $h_\alpha(\zeta) = \lambda \times \varepsilon + \zeta(*) \times \alpha + \zeta$ .

Second:  $(I, J) = (\lambda \times \zeta(*), \zeta(*))$  is a reasonable  $(\mu, \alpha(*))$ -base

Again clause (a) of Definition 6.10(3) holds so we deal with clause (b).

So assume  $e \in \mathbf{e}(J, I)$  and  $h_1, h_2 \in \text{inc}_J(I)$  and  $(h_1, h_2)$  is just a weak  $e$ -pair and  $\mathcal{V} \subseteq \text{Dom}(e)/e$ . Let  $u = \text{Rang}(h_1) \cup \text{Rang}(h_2)$ . For  $\ell = 1, 2$  let  $h_\ell^* \in \text{inc}_J(I)$  be  $h_\ell^*(\zeta) = \text{otp}(u \cap h_\ell(\zeta))$ , so  $\text{Rang}(h_\ell^*) \subseteq \xi(*) := \text{otp}(u) \leq \zeta(*) \times 3$ .

[Why? If  $\zeta(*)$  is finite this is trivial, so assume  $\zeta(*) \geq \omega$ . Let  $n < \omega$  and  $\alpha$  be such that  $\omega^\alpha n \leq \zeta(*) < \omega^\alpha(n+1)$ , so  $\alpha \geq 1, n \geq 1$ . As  $\omega^\alpha$  is additively indecomposable  $\text{otp}(u) \leq \omega^\alpha(2n+1)$ , alternatively use natural sums [MiRa65] which gives a better bound  $\zeta(*) \oplus \zeta(*)$ , [actually  $< \mu^+$  suffices using  $\zeta(*) < \mu^+$  large enough below, still.]

For  $\ell = 1, 2, 3$  we define  $h'_\ell \in \text{inc}_J(I)$  as follows:

- (a) if  $\zeta \in \zeta(*) \setminus \text{Dom}(e)$  then  $h'_\ell(\zeta) = (\zeta(*) \times 4) \times \zeta$
- (b) if  $\zeta \in \text{Dom}(e)$  and  $\varepsilon = \min(\zeta/e)$  and  $\zeta/e \in \mathcal{Y}$  then
  - ( $\alpha$ ) if  $\ell = 3$  then  $h'_\ell(\zeta) = (\zeta(*) \times 4) \times \varepsilon + \zeta(*) \times 3 + \zeta$
  - ( $\beta$ ) if  $\ell = 1, 2$  then  $h'_\ell(\zeta) = (\zeta(*) \times 4) \times \varepsilon + h_\ell^*(\zeta)$
- (c) if  $\zeta \in \text{Dom}(e)$  and  $\varepsilon = \min(\zeta/e)$  and  $\zeta/e \notin \mathcal{Y}$  then
  - ( $\alpha$ ) if  $\ell = 3$  then  $h'_\ell(\zeta) = (\zeta(*) \times 4) \times \varepsilon + \zeta$
  - ( $\beta$ ) if  $\ell = 1, 2$  then  $h'_\ell(\zeta) = (\zeta(*) \times 4) \times \varepsilon + \zeta(*) + h_\ell^*(\zeta)$ .

Now check.

2) First:  $(I, J) = (I_{\mu, \lambda \times \zeta(*)}^{\text{lin}}, I_{\mu, \zeta(*)}^{\text{lin}})$  is a wide  $(\lambda, \mu, \alpha(*))$ -base.

Note that  $P_1^J = w$  because  $\zeta(*) < \mu^+$  and  $P_1^I = \{\alpha \in I : \text{cf}(\alpha) > \mu\}$ . As above clause (a) of the Definition 6.10 holds so we deal with clause (b).

Let

$$u = \{\zeta < \zeta(*) : \zeta \in \text{Dom}(e) \text{ is minimal in } \zeta/e \text{ or } \zeta \in \zeta(*) \setminus \text{Dom}(e)\}.$$

Clearly  $u$  is a closed subset of  $\zeta(*)$  and  $0 \in u$ .

Given  $\zeta < \zeta(*)$  let  $\varepsilon_\zeta := \max(u \cap (\zeta + 1))$ , clearly well defined by the choice of  $u$  and  $\varepsilon_\zeta \leq \zeta$ .

For every  $\alpha < \lambda$  we define  $h_\alpha \in \text{inc}_J(I)$  as follows:

We define  $h_\alpha(\zeta)$  by induction on  $\zeta < \zeta(*)$  such that  $h_\alpha(\zeta) < \lambda \times (\varepsilon_\zeta + 1)$ .

Case A: for  $\zeta \in \zeta(*) \setminus \text{Dom}(e)$

Subcase A1:  $\zeta \in P_1^J$

Let  $h_\alpha(\zeta)$  be  $\lambda \times \varepsilon_\zeta + \mu^+$ .

Subcase A2:  $\zeta \in P_0^J$  and  $\zeta = 0$

Let  $h_\alpha(\zeta) = 0$ .

Subcase A3:  $\zeta \in P_0^J, \zeta = \xi + 1$

Let  $h_\alpha(\zeta) = h_\alpha(\xi) + 1$ .

Subcase A4:  $\zeta \in P_0^J, \zeta$  is a limit ordinal,  $\zeta = \sup(u \cap \zeta)$

Let  $h_\alpha(\zeta) = \lambda \times \varepsilon_\zeta$  which is equal to  $\cup\{h_\alpha(\zeta') : \zeta' < \zeta\}$ .

Subcase A5:  $\zeta \in P_0^J, \zeta$  is a limit ordinal and  $\xi = \sup(u \cap \zeta) < \zeta$ .

So  $(\xi + 1)/e$  is an end-segment of  $\zeta$ , but this is impossible by 6.7(2)(c).

Case B:  $\zeta \in \text{Dom}(e)$ :

Subcase A1:  $\zeta = \min(\zeta/e)$  hence  $\zeta \in P_1^J$  (see 6.7(0)(b))

Let  $h_\alpha(\zeta) = \lambda \times \varepsilon_\zeta + \mu^+ \times \zeta(*) \times \alpha + \mu^+$ .

Subcase A2:  $\zeta \in P_0^J$  hence  $\zeta > \min(\zeta/e)$

Let  $h_\alpha(\zeta) = \cup\{h_\alpha(\zeta') + 1 : \zeta' < \zeta\}$ .

Subcase A3:  $\zeta \in P_1^J$  and  $\zeta > \min(\zeta/e)$

Let  $h_\alpha(\zeta) = \cup\{h_\alpha(\zeta') : \zeta' < \zeta\} + \mu^+$ .

So clearly we can show by induction on  $\zeta < \zeta(*)$  that:

$$h_\alpha(\zeta) < \lambda \times \varepsilon_\zeta + \mu^+ \times \zeta(*) \times (\alpha 2 + 2).$$

Now check.

Also recalling  $\mu^+ < \lambda$  clearly for  $\alpha < \lambda, \zeta < \zeta(*)$  we have  $h_\alpha(\zeta) < \lambda \times \varepsilon_\zeta + \lambda$ .

Now check.

Second  $(I_{\mu, \lambda \times \zeta(*)}^{\text{lin}}, I_{\mu, \zeta(*)}^{\text{lin}})$  is a reasonable  $(\mu, \alpha(*))$ -base

Combine the proof of “first” with the parallel proof in part (1).  $\square_{6.12}$

**6.13 Definition.** 1) Let  $I, J \in K_{\tau_{\alpha(*)}}^{\text{lin}}$ . We say that  $\mathcal{E}$  is an invariant  $(I, J)$ -equivalence relation when:

- (a)  $\mathcal{E}$  is an equivalence relation on  $\text{inc}_J(I)$ , so  $\mathcal{E}$  determines  $I$  and  $J$
- (b) if  $h_1, h_2, h_3, h_4 \in \text{inc}_J(I)$  and  $\text{tp}_{\text{qf}}(h_1, h_2; I) = \text{tp}_{\text{qf}}(h_3, h_4; I)$  then  $h_1 \mathcal{E} h_2 \Leftrightarrow h_3 \mathcal{E} h_4$ .

2) We add non-trivial when:

- (c) if  $\text{eq}(h_1, h_2) = \{t \in J : h_1(t) = h_2(t)\}$  is co-finite then  $h_1 \mathcal{E} h_2$
- (d) there are  $h_1, h_2 \in \text{inc}_J(I)$  such that  $\neg(h_1 \mathcal{E} h_2)$ .

3) Let  $J, I_1, I_2 \in K_{\tau_{\alpha(*)}}^{\text{lin}}$ . Then  $I_1 \leq_J^1 I_2$  means that:

- (a)  $I_1 \subseteq I_2$
- (b) for every  $h_1, h_2, h_3 \in \text{inc}_J(I_2)$  we can find  $h'_1, h'_2, h'_3 \in \text{inc}_J(I_1)$  such that  $\text{tp}_{\text{qf}}(h'_1, h'_2, h'_3; I_1) = \text{tp}_{\text{qf}}(h_1, h_2, h_3; I_2)$ .



**6.14 Claim.** Assume  $J, I_1, I_2 \in K_{\tau_{\alpha(*)}}^{\text{lin}}$ .

- 1) If  $I_1 \subseteq I_2$ ,  $\mathcal{E}$  is an invariant  $(I_2, J)$ -equivalence relation then  $\mathcal{E} \upharpoonright \text{inc}_J(I_1)$  is an invariant  $(I_1, J)$ -equivalence relation.
- 2) If  $I_1 <_J^1 I_2$  and  $\mathcal{E}_1$  is an invariant  $(I_1, J)$ -equivalence relation then there is one and only one invariant  $(I_2, J)$ -equivalence relation  $\mathcal{E}_2$  such that  $\mathcal{E}_2 \upharpoonright \text{inc}_J(I_1) = \mathcal{E}_1$ .
- 3) Assume  $e \in \mathbf{e}(J)$  and  $\mathcal{Y} \subseteq \text{Dom}(e)/e$ . If  $(h'_1, h'_2)$  is a strict  $(e, \mathcal{Y})$ -pair for  $(I_1, J)$  and  $(h''_1, h''_2)$  is a strict  $(e, \mathcal{Y})$ -pair for  $(I_2, J)$  then  $\text{tp}_{\text{qf}}(h'_1, h'_2; I_1) = \text{tp}_{\text{qf}}(h''_1, h''_2; I_2)$ .
- 4) Assume  $\alpha(*) = 1, J = \zeta(*), I_\ell = \beta_\ell$  with the usual order (for  $\ell = 1, 2$ ),  $\mu \leq \zeta(*) < \mu^+$  and  $\mu^+ \leq \beta_1 \leq \beta_2$ . Then  $I_1 <_J^1 I_2$  (see Definition 6.13(3)).
- 5) Assume  $\alpha(*) = 2, J = I_{\mu, \zeta(*), w}^{\text{lin}}, I_\ell = I_{\mu, \beta_\ell}^{\text{lin}}$  for  $\ell = 1, 2$  and  $\mu^{++} \leq \beta_1 \leq \beta_2$ . Then  $I_1 <_J^1 I_2$  (see Definition 6.13(3)).

*Proof.* 1) Obvious.

2) We define

$$\begin{aligned} \mathcal{E}_2^* = \{ (h_1, h_2) : & h_1, h_2 \in \text{inc}_J(I_2) \text{ and for some} \\ & h'_1, h'_2 \in \text{inc}_J(I_1) \text{ we have} \\ & \text{tp}_{\text{qf}}(h'_1, h'_2; I_1) = \text{tp}_{\text{qf}}(h_1, h_2; I_2) \text{ and} \\ & h'_1 \mathcal{E}_1 h'_2 \}. \end{aligned}$$

Now

(\*)<sub>1</sub>  $\mathcal{E}_2^*$  is a set of pairs of members of  $\text{inc}_J(I_2)$ .

[Why? By its definition]

(\*)<sub>2</sub>  $h_1 \mathcal{E}_2^* h_2$  if  $h_1 \in \text{inc}_J(I_2)$ .

[Why? Let  $h' \in \text{inc}_J(I_1)$  so clearly  $h' \mathcal{E}_1 h'$  and  $\text{tp}_{\text{qf}}(h', h'; I_1) = \text{tp}_{\text{qf}}(h, h; I_2)$ ]

(\*)<sub>3</sub>  $\mathcal{E}_2^*$  is symmetric.

[Why? As  $\mathcal{E}_1$  is.]

(\*)<sub>4</sub>  $\mathcal{E}_2^*$  is transitive.

[Why? Assume  $h_1 \mathcal{E}_2^* h_2$  and  $h_2 \mathcal{E}_2^* h_3$  and let  $h'_1, h'_2 \in \text{inc}_J(I_1)$  witness  $h_1 \mathcal{E}_2^* h_2$  and  $h'_2, h'_3 \in \text{inc}_J(I_1)$  witness  $h_2 \mathcal{E}_2^* h_3$ .

Apply clause (b) of part (3) of Definition 6.13 to  $(h_1, h_2, h_3)$  so there are  $g_1, g_2, g_3 \in \text{inc}_J(I_1)$  such that  $\text{tp}_{\text{qf}}(g_1, g_2, g_3; I_1) = \text{tp}_{\text{qf}}(h_1, h_2, h_3; I_2)$ . Now  $h'_1 \mathcal{E}_1 h'_2$  by the

choice of  $(h'_1, h'_2)$  and  $\text{tp}_{\text{qf}}(g_1, g_2; I_1) = \text{tp}_{\text{qf}}(h_1, h_2; I_2) = \text{tp}_{\text{qf}}(h'_1, h'_2; I_1)$  so as  $\mathcal{E}_1$  is invariant we get  $g_1 \mathcal{E}_1 g_2$ . Similarly  $g_2 \mathcal{E}_1 g_3$ , so as  $\mathcal{E}_1$  is transitive we have  $g_1 \mathcal{E}_1 g_3$ . But clearly  $\text{tp}_{\text{qf}}(g_1, g_3; I_1) = \text{tp}_{\text{qf}}(h_1, h_3; I_2)$  hence  $g_1, g_2$  witness that  $h_1 \mathcal{E}_2 h_3$  is as required.]

(\*)<sub>5</sub>  $\mathcal{E}_2^*$  is invariant.

[Why? See its definition.]

(\*)<sub>6</sub>  $\mathcal{E}_2^* \upharpoonright \text{inc}_I(I_1) = \mathcal{E}_1$ .

[Why? By the way  $\mathcal{E}_2^*$  is defined and  $\mathcal{E}_1$  being invariant.]

So together  $\mathcal{E}_2^*$  is as required. The uniqueness (i.e. if  $\mathcal{E}_2$  is an invariant equivalent relation on  $\text{inc}_J(I)$  such that  $\mathcal{E}_2 \upharpoonright \text{inc}_J(I_1) = \mathcal{E}_1$  then  $\mathcal{E}_2 = \mathcal{E}_2^*$ ) is also easy.

3) Straight.

4) See<sup>6</sup> the proof of “Second” in the proof of 6.12(1).

5) Combine<sup>7</sup> the proof of part (4) and of “First” in the proof of 6.12(2).  $\square_{6.14}$

Below mostly it suffices to consider  $\mathcal{D}_{\mathcal{E}, e}$ .

**6.15 Definition.** 1) Let  $\mathcal{E}$  be an invariant  $(I, J)$ -equivalence relation; we define

$$\mathcal{D}_{\mathcal{E}} = \{u \subseteq J : \text{if } h_1, h_2 \in \text{inc}_J(I) \text{ satisfies } \text{eq}(h_1, h_2) \supseteq u \text{ then } h_1 \mathcal{E} h_2\}$$

recalling

$$\text{eq}(h_1, h_2) := \{t \in J : h_1(t) = h_2(t)\}.$$

2) If in addition  $e \in \mathbf{e}(J, I)$  then we let

$$\mathcal{D}_{\mathcal{E}, e} = \{u \subseteq \text{Dom}(e)/e : \text{if } h_1, h_2 \in \text{inc}_J(I) \text{ and } (h_1, h_2) \text{ is an } (e \upharpoonright (\text{Dom}(e) \setminus \text{set}(u))\text{-pair then } h_1 \mathcal{E} h_2\}.$$

---

<sup>6</sup>Actually instead “ $\mu^+ \leq \beta_1$ ” it suffice to have  $\zeta(*) \times 4 \leq \beta_1$  because if  $\zeta(*) = \sum_{i < \gamma} \zeta_i$  then

$\sum_{i < \gamma} \zeta_i \times 4 \leq \zeta(*) \times 4$  or just the natural sum  $\zeta(*) \oplus \zeta(*) \oplus \zeta(*)$ .

<sup>7</sup>Here  $(\mu^+ + 1) \times (\zeta(*) \times 4)$  will suffice.

**6.16 Claim.** Assume  $I, J \in K_{\tau_{\alpha(*)}}^{\text{lin}}$  and  $(I, J)$  is reasonable (see Definition 6.10(3), (6))

and  $\mathcal{E}$  is an invariant  $(I, J)$ -equivalence relation.

1) For  $u \subseteq J$  such that  $e_{J,u} \in \mathbf{e}(J, I)$  we have:  $u \in \mathcal{D}_{\mathcal{E}}$  iff  $h_1 \mathcal{E} h_2$  for every  $e_{J,u}$ -pair  $(h_1, h_2)$  iff  $h_1 \mathcal{E} h_2$  for some  $e_{J,u}$ -pair  $(h_1, h_2)$ ; see Definition 6.9(1).

2) Assume  $e \in \mathbf{e}(J, I)$ , then for any  $u \subseteq \text{Dom}(e)/e$  we have:  $u \in \mathcal{D}_{\mathcal{E},e}$  iff  $h_1 \mathcal{E} h_2$  for any  $(e \upharpoonright \text{set}(u))$ -pair iff  $h_1 \mathcal{E} h_2$  for some  $(e \upharpoonright \text{set}(u))$ -pair.

3) If  $e \in \mathbf{e}(J, I)$  and  $u_1, u_2 \subseteq \text{Dom}(e)/e$  then we can find  $h_1, h_2, h_3 \in \text{inc}_J(t)$  such that  $(h_1, h_2)$  is a strict  $(e \upharpoonright \text{set}(u_1))$ -pair,  $(h_2, h_3)$  is a strict  $(e \upharpoonright \text{set}(u_2))$  pair and  $(h_1, h_3)$  is a strict  $(e \upharpoonright (\text{set}(u_1 \cup u_2)))$ -pair.

4) Assume  $e \in \mathbf{e}(J, I)$  and that in clause (b) of Definition 6.10(3) we allow  $(h_1, h_2)$  to be a weak  $e$ -pair, then for any  $u \subseteq \text{Dom}(e)/e$  we have:  $\text{Dom}(e) \setminus u \in \mathcal{D}_{\mathcal{E},e}$  iff  $h_1 \mathcal{E} h_2$  for every weak  $e$ -pair  $(h_1, h_2)$ .

*Proof.* 1) Like part (2).

2) In short, by transitivity of equivalence and the definitions + mixing, but we elaborate.

The “first implies the second” holds by Definition 6.15(2) and “the second implies the third” holds trivially as there is such a pair  $(h_1, h_2)$  by the assumption  $e \in \mathbf{e}(J, I)$ . So it is enough to prove “the third implies the first”; hence suppose that  $g_1 \mathcal{E} g_2$ , where  $(g_1, g_2)$  is an  $e_1 := e \upharpoonright \text{set}(u)$ -pair (recalling that  $e_1 \in \mathbf{e}(J, I)$  by 6.8(8)), and let  $(h_1, h_2)$  be an  $e_1$ -pair, we need to show that  $h_1 \mathcal{E} h_2$ . By Definition 6.6(2B) for some sets  $\mathcal{Y}_g, \mathcal{Y}_h \subseteq \text{Dom}(e_1)/e_1$  the pair  $(g_1, g_2)$  is a strict  $(e_1, \mathcal{Y}_g)$ -pair and the pair  $(h_1, h_2)$  is a strict  $(e_1, \mathcal{Y}_h)$ -pair. Recalling clause (b) of 6.10(3) there are  $g'_1, g'_2, g'_3$  and  $\mathcal{Y}$  such that:

- (\*)<sub>1</sub> (a)  $g'_\ell \in \text{inc}_J(I)$  for  $\ell = 1, 2, 3$
- (b)  $\text{tp}_{\text{qf}}(g_1, g_2) = \text{tp}_{\text{qf}}(g'_1, g'_2)$
- (c)  $\mathcal{Y} \subseteq \text{Dom}(e_1)/e_1$
- (d)  $(g'_1, g'_3)$  and  $(g'_2, g'_3)$  are strict  $(e_1, \mathcal{Y})$ -pairs.

Now for each  $s \in \text{Dom}(e_1)$ , we can find a permutation  $\bar{\ell}_s = (\ell_{s,1}, \ell_{s,2}, \ell_{s,3})$  of  $\{1, 2, 3\}$  such that  $I \models g'_{\ell_{s,1}}(s) < g'_{\ell_{s,2}}(s) < g'_{\ell_{s,3}}(s)$ . By (\*)<sub>1</sub>(d) and (\*)<sub>1</sub>(b) and  $(g_1, g)$  being an  $e_1$ -pair, clearly  $\bar{\ell}_s$  depends only on  $s/e_1$  and every member of  $\{(g'_{\ell_{s,1}}(t) : t \in s/e_1)\}$  is below every member of  $\{(g'_{\ell_{s,2}}(t) : t \in s/e_1)\}$  and similarly for the pair  $(g'_{\ell_{s,2}}, g'_{\ell_{s,3}})$ . Now we can find  $(g''_1, g''_2, g''_3)$  such that:

- (\*)<sub>2</sub> (a)  $g''_\ell \in \text{inc}_J(I)$  for  $\ell = 1, 2, 3$
- (b)  $(g''_1, g''_2)$  is a strict  $(e_1, \mathcal{Y}_h)$
- (c)  $(g''_1, g''_3)$  and  $(g''_2, g''_3)$  are strict  $(e_1, \mathcal{Y}_g)$ -pairs

[Why? We do the choice for each  $s/e_1$  separately such that:  $\{g_1'' \upharpoonright (s/e_1), g_2'' \upharpoonright (s/e_1), g_3'' \upharpoonright (s/e_1)\} = \{g_1' \upharpoonright (s/e_1), g_2' \upharpoonright (s/e_1), g_3' \upharpoonright (s/e_1)\}$ .]

Clearly  $\text{tp}_{\text{qf}}(g_1'', g_3''; I) = \text{tp}_{\text{qf}}(g_1, g_2; I) = \text{tp}_{\text{qf}}(g_2'', g_3''; I)$  so as  $\mathcal{E}$  is invariant and  $g_1 \mathcal{E} g_2$  clearly  $g_1'' \mathcal{E} g_3'' \wedge g_2'' \mathcal{E} g_3''$  which implies  $g_1'' \mathcal{E} g_2''$ . For  $\mathcal{Y}' = \mathcal{Y}_h$  by clause (b) of  $(*)_2$  we conclude that  $\text{tp}_{\text{qf}}(g_1'', g_2''; I) = \text{tp}_{\text{qf}}(h_1, h_2; I)$  so as  $\mathcal{E}$  is invariant we are done.

3),4) Similarly.  $\square_{6.16}$

**6.17 Claim.** Assume  $I, J \in K_{\tau_{\alpha(*)}}^{\text{lin}}$  and  $\mathcal{E}$  is an invariant  $(I, J)$ -equivalence relation.

0) If  $e \in \mathbf{e}(J, I)$  and  $\mathcal{E}$  is non-trivial then  $\mathcal{D}_{\mathcal{E}, e}$  contains all co-finite subsets of  $\text{Dom}(e)/e$ .

1) If the pair  $(I, J)$  is reasonable and  $e \in \mathbf{e}(I, J)$  then  $\mathcal{D}_{\mathcal{E}, e}$  is a filter on  $\text{Dom}(e)/e$  but possibly  $\emptyset \in \mathcal{D}_{\mathcal{E}, e}$ .

2) (a)  $\mathcal{D}_{\mathcal{E}}$  is a filter on  $J$

(b) if  $\mathcal{E}$  is non-trivial then all cofinite subsets of  $J$  belongs to  $\mathcal{D}_{\mathcal{E}}$  but  $\emptyset \notin \mathcal{D}_{\mathcal{E}}$ .

*Proof.* 0) Easy, see Definition 6.13(2).

1) By 6.16(2) and 6.16(3).

2) Trivial by Definition 6.15(1).  $\square_{6.17}$

**6.18 Main Claim.** Assume

(a)  $I, J \in K_{\tau_{\alpha(*)}}^{\text{lin}}$

(b)  $\mathcal{E}$  is an invariant  $(I, J)$ -equivalence relation

(c)  $(I, J)$  is a reasonable  $(\mu, \alpha(*))$ -base which is a wide  $(\lambda, \mu, \alpha(*))$ -base

(d)  $e \in \mathbf{e}(J, I)$

(e)  $g$  is a function from  $\text{Dom}(e)/e$  into some cardinal  $\theta$

(f)  $\mathcal{D}^* = \{Y \subseteq \theta : g^{-1}(Y) \in \mathcal{D}_{\mathcal{E}, e}\}$  is a filter, i.e.,  $\emptyset \notin \mathcal{D}^*$ .

Then  $\mathcal{E}$  has at least  $\chi := \lambda^\theta / \mathcal{D}^*$  equivalence classes.

*Proof.* Let  $\langle f_\alpha : \alpha < \chi \rangle$  be a set of functions from  $\theta$  to  $\lambda$  exemplifying  $\chi := \lambda^\theta / \mathcal{D}^*$  so  $\alpha \neq \beta \Rightarrow \{i < \theta : f_\alpha(i) = f_\beta(i)\} \notin \mathcal{D}^*$ .

Let  $\langle h_\zeta : \zeta < \lambda \rangle$  exemplify the pair  $(I, J)$  being a wide  $(\lambda, \mu, \alpha(*))$ -base, see Definition 6.10(4), so  $h_\zeta \in \text{inc}_J(I)$ .

Lastly for each  $\alpha < \chi$  we define  $h^\alpha \in \text{inc}_J(I)$  as follows:  
 $h^\alpha(t)$  is:  $h_0(t)$  if  $t \in J \setminus \text{Dom}(e)$

$h_{f_\alpha(g(t/e))}(t) \text{ if } t \in \text{Dom}(e).$

Now

(\*)<sub>1</sub>  $h^\alpha$  is a function from  $J$  to  $I$ .

[Why? Trivially recalling each  $h_\zeta$  is.]

(\*)<sub>2</sub>  $h^\alpha$  is increasing.

[Why? Let  $s <_J t$  and we split the proof to cases.

If  $s, t \in J \setminus \text{Dom}(e)$  use “ $h_0 \in \text{inc}_J(I)$ ”.

If  $s \in J \setminus \text{Dom}(e)$  and  $t \in \text{Dom}(e)$ , then  $h^\alpha(t) = h_{f_\alpha(g(t/e))}(t)$ ,  $h^\alpha(s) = h_0(s) = h_{f_\alpha(g(t/e))}(s)$  because  $\langle h_\alpha \upharpoonright (J \setminus \text{Dom}(e)) : \alpha < \lambda \rangle$  is constant (recalling  $(h_0, h_\alpha)$  is an  $e$ -pair (for  $\alpha > 0$ )), so as  $h_{f_\alpha(g(t/e))} \in \text{inc}_J(I)$  we are done.

If  $s \in \text{Dom}(e)$ ,  $t \in J \setminus \text{Dom}(e)$ , the proof is similar.

If  $s, t \in \text{Dom}(e)$ ,  $s/e \neq t/e$ , we again use Definition 6.6(2B), clause (b)( $\beta$ ) of Definition 6.10(4).

Lastly, if  $s, t \in \text{Dom}(e)$ ,  $s/e = t/e$  we get  $g(s/e) = g(t/e)$  hence  $f_\alpha(g(s/e)) = f_\alpha(g(t/e))$  call it  $\gamma$  so  $h^\alpha(s) = h_\gamma(s)$ ,  $h^\alpha(t) = h_\gamma(t)$  and of course  $h_\gamma \in \text{inc}_J(I)$  hence  $h_\gamma(s) <_I h_\gamma(t)$  so necessarily  $h^\alpha(s) <_I h^\alpha(t)$  as required. So (\*)<sub>2</sub> holds.]

(\*)<sub>3</sub>  $h^\alpha \in \text{inc}_J(I)$ .

[Why? Clearly if  $i < \alpha(*)$  and  $t \in P_i^J$  then  $(\forall \beta < \lambda) h_\beta(t) \in P_i^J$  hence  $\alpha < \chi \Rightarrow h_{f_\alpha(g(t/e))}(t) \in P_i^J$  which means  $\alpha < \chi \Rightarrow h^\alpha(t) \in P_i^J$ ; so recalling (\*)<sub>2</sub>, clause (a) of Definition 6.3(2) holds. We should check clauses (b),(c) of Definition 6.3(2) which is done as in the proof of 6.7 and of (\*)<sub>2</sub> above.]

(\*)<sub>4</sub> if  $\alpha < \beta$  and we let  $u = u_{\alpha,\beta} := \cup\{g^{-1}(\zeta) : \zeta < \theta \text{ and } f_\alpha(\zeta) \neq f_\beta(\zeta)\}$  so  $u \subseteq \text{Dom}(e)/e$  then  $(h^\alpha, h^\beta)$  is a  $(e \upharpoonright \text{set}(u))$ -pair.

[Why? Case 1: If  $s \in J \setminus \text{Dom}(e)$  then  $h^\alpha(s) = h_0(s) = h^\beta(s)$ .

Case 2: If  $s \in \text{Dom}(e) \setminus \text{set}(u)$  then  $h^\alpha(s) = h_{f_\alpha(g(s/e))}(s) = h_{f_\beta(g(s/e))}(s) = h^\beta(s)$ .

Case 3: If  $s, t \in \text{set}(u)$ ,  $s/e \neq t/e$ ,  $s <_J t$  then  $h^\alpha(s) <_I h^\beta(t) \wedge h^\beta(s) <_I h^\alpha(t)$  because

Subcase 3A: If  $f_\alpha(g(s/e)) = f_\beta(g(t/e))$  we use  $h_{f_\alpha(g(t/e))} \in \text{inc}_J(I)$  hence

$$h^\alpha(s) = h_{f_\alpha(g(s/e))}(s) <_I h_{f_\alpha(g(s/e))}(t) = h_{f_\beta(g(t/e))}(t) = h^\beta(t)$$

and similarly  $h^\beta(s) <_I h^\alpha(t)$ .

Subcase 3B:  $f_\alpha(g(s/e)) \neq f_\beta(g(t/e))$  we use “ $(h_{f_\alpha(g(s/e))}, h_{f_\beta(g(t/e))})$  is an  $e$ -pair”.

Case 4: And lastly, if  $s, t \in \text{set}(u)$ ,  $s/e = t/e$  and  $s <_J t$  then  $h^\alpha(t) <_I h^\beta(s) \equiv (s/e \in u) \equiv h^\alpha(s) <_I h^\beta(t)$ .

Why? Recalling  $f_\alpha(g(s/e)) \neq f_\beta(g(t/e))$  as  $s, t \in \text{set}(u)$  by the definition of  $u$ , see  $(*)_4$  and we just use “ $(h_{f_\alpha(g(s/e))}, h_{f_\beta(g(s/e))})$ ” is an  $e$ -pair and clause (c)' of Definition 6.6.]

$(*)_5$  if  $\alpha < \beta$  then  $u_{\alpha,\beta} \neq \emptyset \bmod \mathcal{D}_{\mathcal{E},e}$ .

[Why? By the choice of  $\langle f_\alpha : \alpha < \lambda \rangle$ .]

$(*)_6$  if  $\alpha < \beta$  then  $h^\alpha, h^\beta$  are not  $\mathcal{E}$ -equivalent.

[Why? By  $(*)_4 + (*)_5$  and 6.16(2).]

Together we are done. □<sub>6.18</sub>

**6.19 Claim.** Assume  $\mathcal{E}$  is an invariant  $(I, J)$ -equivalence relation,  $I, J$  are well ordered and  $|\text{inc}_J(I)/\mathcal{E}| \geq \lambda = \text{cf}(\lambda) > \mu = |I| > |2 + \alpha(*)|^{J|}$ . Then for some  $e \in \mathbf{e}(I, J)$  there is an ultrafilter  $\mathcal{D}$  on  $\text{Dom}(e)/e$  extending  $\mathcal{D}_{\mathcal{E},e}$  which is not principal.

*Remark.* This is close to [Sh 620, §7].

*Proof.* Without loss of generality as linear orders,  $J$  is  $\zeta(*)$  and  $I$  is  $\xi(*) \in [\mu, \mu^+)$ .

Toward contradiction assume the conclusion fails. Let  $g$  be a one-to-one function from  $\mu$  onto  $[\xi(*)]^{<\aleph_0}$  and  $\chi$  be large enough and  $\kappa = |J|$  and  $\partial = |2 + \alpha(*)|^{J|}$  so  $\partial^\kappa = \partial$ .

We now choose  $\langle N_\eta : \eta \in {}^n\mu \rangle$  by induction on  $n < \omega$  such that

- ⊗<sub>1</sub> (a)  $N_\eta \prec (\mathcal{H}(\chi), \in)$
- (b)  $\|N_\eta\| = \partial$  and  $\partial + 1 \subseteq N_\eta$
- (c)  $A \subseteq N_\eta \wedge |A| \leq \kappa \Rightarrow A \in N_\eta$
- (d)  $I, J$  and  $g$  as well as  $\eta$  belong to  $N_\eta$
- (e)  $\nu \triangleleft \eta \Rightarrow N_\nu \in N_\eta$  (hence  $N_\nu \subseteq N_\eta$  so  $N_\nu \prec N_\eta$ ).

There is no problem to do this. Now it suffices to prove that for every  $h \in \text{inc}_J(I)$ , for some  $h' \in \cup\{N_\eta : \eta \in {}^{\omega>}\mu\} \cap \text{inc}_J(I)$  we have  $h \mathcal{E} h'$ .

Fix  $h_* \in \text{inc}_J(I)$  such that  $h_* \notin \cup\{h/\mathcal{E} : h \in \text{inc}_J(I) \cap N_\eta \text{ for some } \eta \in {}^{\omega>}\mu\}$  and for each  $\eta \in {}^{\omega>}\mu$  we define  $\bar{\alpha}_\eta, e_\eta$  as follows:

- ⊗<sub>2</sub> (a)  $\bar{\alpha}_\eta = \langle \alpha_{\eta,t} : t \in J \rangle$

- (b)  $\alpha_{\eta,t} = \min((\xi(*) + 1) \cap N_\eta \setminus h_*(t))$
- (c)  $e_\eta := \{(s, t) : s, t \in J \text{ and } \alpha_{\eta,s} = \alpha_{\eta,t} \text{ and } \alpha_{\eta,s} > h_*(s) \text{ and } \alpha_{\eta,t} > h_*(t)\}$
- (d) for  $\alpha \in N_\eta$  let  $X_{\eta,\alpha} := \{t \in J : \alpha_{\eta,t} = \alpha > h_*(t)\}$ .

Note

$$(*)_1 \quad \bar{\alpha}_\eta \in N_\eta.$$

[Why? As  $[N_\eta]^{\leq \kappa} \subseteq N_\eta$  and  $|J| = \kappa$  and  $\alpha_{\eta,t} \in N_\eta$  for every  $t \in J$ .]

- (\*)<sub>2</sub> (a)  $e_\eta \in \mathbf{e}(J)$ , i.e.  $e_\eta$  is an equivalence relation on some subset of  $J$  with each equivalence class a convex subset of  $J$ , see Definition 6.6(1)
- (b)  $\langle X_{\eta,\alpha} : \alpha \in \{\alpha_{\eta,t} : t \in \text{Dom}(e_\eta)\} \text{ hence } X_{\eta,\alpha} \neq \emptyset \rangle$  list the  $e_\eta$ -equivalence classes.

[Why? Think.]

$$(*)_3 \quad h_\eta := h_* \upharpoonright (J \setminus \text{Dom}(e_\eta)) \in N_\eta.$$

[Why? By the definition of  $e_\eta$  we have  $t \in J \wedge t \notin \text{Dom}(e_\eta) \Rightarrow h_*(t) \in N_\eta$  and recall  $[N_\eta]^{\leq \kappa} \subseteq N_\eta$ .]

$$(*)_4 \quad \text{if } t \in \text{Dom}(e_\eta) \text{ then } \text{cf}(\alpha_{\eta,t}) > \partial.$$

[Why? As  $\alpha_{\eta,t} \in N_\eta \prec (\mathcal{H}(\chi), \in)$  if  $\text{cf}(\alpha_{\eta,t}) = \theta \leq \partial$  then there is a cofinal set  $B$  of  $\alpha_{\eta,t}$  of cardinality  $\theta$  in  $N_\eta$  but  $\theta \leq \partial + 1 \subseteq N_\eta$  therefore  $B \subseteq N_\eta$ . In particular as  $h_*(t) < \alpha_{\eta,t}$  there is  $\beta \in B$  so that  $h_*(t) < \beta$ , but this contradicts the choice of  $\alpha_{\eta,t}$ .]

$$(*)_5 \quad e_\eta \in \mathbf{e}(J, I).$$

[Why? Choose  $h' \in \text{inc}_J(I) \cap N_\eta$  similar enough to  $h_*$ , specifically:  $t \in J \setminus \text{Dom}(e_\eta) \Rightarrow h'(t) = h_*(t)$  and  $t \in \text{Dom}(e_\eta) \Rightarrow \sup\{\alpha_{\eta,s} : s \in J, s <_J t \text{ and } s \notin t/e_\eta\} < h'(t) < \alpha_{\eta,t}$ . The point being that  $\sup\{\alpha_{\eta,s} : s \in J, s <_J t \text{ and } s \notin t/e_\eta\} \in N_\eta$ . Now  $(h', h_*)$  is a strict  $e$ -pair.]

- (\*)<sub>6</sub> there is  $\ell_\eta < \omega$  and a finite sequence  $\langle \beta_{\eta,\ell} : \ell < \ell_\eta \rangle$  of members of  $\text{Rang}(\bar{\alpha}_\eta \upharpoonright \text{Dom}(e_\eta))$  so  $X_{\eta,\beta_{\eta,\ell}} \in \text{Dom}(e_\eta)/e_\eta$  for  $\ell < \ell_\eta$  such that  $\cup\{X_{\eta,\beta_{\eta,\ell}} : \ell < \ell_\eta\} \in \mathcal{D}_{\mathcal{E}, e_\eta}$ .

[Why? Otherwise there is an ultrafilter as desired, but toward contradiction we have assumed this does not occur; in trying to get generalizations we should act differently.]

Now we choose  $(\eta_n, h_n)$  by induction on  $n < \omega$  such that

- (a)  $\eta_n \in {}^n\mu$
- (b) if  $n = m + 1$  then  $\eta_m = \eta_n \upharpoonright m$
- (c)  $h_n \in \text{inc}_J(I)$
- (d)  $h_0 = h_*$
- (e) if  $n = m + 1$  then:
  - (α)  $h_n \mathcal{E} h_m$  hence  $h_n \mathcal{E} h_*$  and  $\text{Dom}(e_{\eta_n}) \subseteq \text{Dom}(e_{\eta_m})$
  - (β)  $h_m \upharpoonright (J \setminus \text{Dom}(e_{\eta_m})) \subseteq h_n$
  - (γ)  $(h_m \upharpoonright (\cup \{X_{\eta_m, \beta_{\eta_m, \ell}} : \ell < \ell_{\eta_m}\})) \subseteq h_n$
  - (δ)  $h_n \upharpoonright (\text{Dom}(e_{\eta_m}) \setminus \cup \{X_{\eta_m, \beta_{\eta_m, \ell}} : \ell < \ell_{\eta_m}\})$  belongs to  $N_{\eta_m}$
  - (ε) moreover  $t \in \text{Dom}(e_{\eta_m}) \setminus \cup \{X_{\eta_m, \beta_{\eta_m, \ell}} : \ell < \ell_{\eta_m}\}$  implies  $h_n(t) < h_m(t)$
  - (ζ)  $\ell_{\eta_m} > 0$
- (f)  $Y_{m+1} \subseteq Y_m$  where  $Y_m := \cup \{X_{\eta_m, \beta_{\eta_m, \ell}} : \ell < \ell_{\eta_m}\}$ .

Why can we carry out the construction? For  $n = 0$  we obviously can (choose  $h_0 = h_*$ ). For  $n = m + 1$  first choose  $h'_m \in N_{\eta_m}$  as we choose in the proof of  $(*)_5$ . Now recalling  $\langle X_{\eta_m, \beta_{\eta_m, \ell}} : \ell < \ell_{\eta_m} \rangle$  was chosen in  $(*)_6$ , and define  $h_n$  by  $h_n \upharpoonright (\text{Dom}(e_{\eta_m}) \setminus \cup \{X_{\eta_m, \beta_{\eta_m, \ell}} : \ell < \ell_{\eta_m}\}) = h'_m \upharpoonright (\text{Dom}(e_{\eta_m}) \setminus \cup \{X_{\eta_m, \beta_{\eta_m, \ell}} : \ell < \ell_{\eta_m}\})$  and  $h_n \upharpoonright (J \setminus \text{Dom}(e_{\eta_m})) = h_m \upharpoonright (J \setminus \text{Dom}(e_{\eta_m}))$  and  $h_n \upharpoonright (\cup \{X_{\eta_m, \beta_{\eta_m, \ell}} : \ell < \ell_{\eta_m}\}) = h_m \upharpoonright (\cup \{X_{\eta_m, \beta_{\eta_m, \ell}} : \ell < \ell_{\eta_m}\})$ . Why  $h_n \mathcal{E} h_m$ ? Because

- (i) as in the proof of  $(*)_5$ ,  $(h_n, h_m)$  form a strict  $\ell_{\eta}$ -pair
- (ii) they agree on  $\cup \{X_{\eta_m, \beta_{\eta_m, \ell}} : \ell < \ell_{\eta}\}$
- (iii)  $\{X_{\eta_m, \beta_{\eta_m, \ell}} : \ell < n\} \in \mathcal{D}_{\mathcal{E}, \mathbf{e}_{\eta}}$ .

Lastly, choose  $\eta_n = \eta_m \hat{\ } \langle \gamma_m \rangle$  where  $\gamma_m$  is chosen such that  $g(\gamma_m) = \{\sup(\beta_{\eta_m, \ell} \setminus \sup\{h_m(t) : t \in X_{\beta_{\eta_m, \ell}}\}) : \ell < \ell_{\eta_m}\}$  recalling that  $g$  is a function from  $\mu$  onto  $[\xi(*)]^{<\aleph_0} = [I]^{<\aleph_0}$ .

Now check that  $\eta_n, h_n$  are as required.

Note that this induction never stops in the sense that  $h_n \notin N_{\eta_n}$  recalling the choice of  $h_*$  and  $h_n \mathcal{E} h_*$ . Now  $\mathcal{U}_n := \{\beta_{\eta_n, \ell} : \ell < n_{\eta}\}$  is a finite non-empty set of ordinals, and if  $n = m + 1$ , then easily  $(\forall \ell < \ell_{\eta_n})(\exists k < \ell_{\eta_m})(\beta_{\eta_n, \ell} < \beta_{\eta_m, k})$  because for  $\ell < \ell_{\eta_n}$  letting  $t \in X_{\eta_n, \ell}$  we know that for some  $k \leq \ell_{\eta_m}$  we have  $t \in X_{\eta_m, k}$  and  $\eta_n(m)$  was chosen above such that as  $\gamma_m$ , now  $h_*(t) \leq \gamma_n \in N_{\eta_n}$ ,  $\gamma_m \leq \alpha_{\eta_m, t}$  and the inequality is strict as  $\text{cf}(\alpha_{\eta_m, t}) > 0$ . So  $\langle \max(\mathcal{U}_n) : n < \omega \rangle$  is a decreasing sequence of ordinals, contradiction, so we are done. □<sub>6.19</sub>



6.20 Example: For  $e \in \mathbf{e}(J, I)$ ,  $J \in K_{\tau_{\alpha(*)}}^{\text{lin}}$  and  $I \in K_{\tau_{\alpha(*)}}^{\text{lin}}$  we define  $\mathcal{E}_e^* = \mathcal{E}_{e,I}^*$ ; it is an invariant equivalent relation on  $\text{inc}_J(I)$ , by:  $h_1 \mathcal{E}_{e,I}^* h_2$  iff:

- (a) if  $t \in J \setminus \text{Dom}(e)$  then  $h_1(t) = h_2(t)$
- (b) if  $t \in \text{Dom}(e)$  then  $\text{cnv}_{I,h_1}(t) = \text{cnv}_{I,h_2}(t)$  where  $\text{cnv}_{I,h}(s) :=$  the convex hull (in  $I$ ) of the set  $\{h_1(s)\} \cup \bigcup \{[h_1(s), h_1(t)]_I : s <_J t \text{ and } t \in s/e\} \cup \bigcup \{[h(t), h(s)]_I : t <_J s \text{ and } t \in s/e\}$ .

- 1) If  $J, I \in K_{\tau_{\alpha(*)}}^{\text{lin}}$  are well ordered and  $e = J \times J$  then  $\mathcal{E}_{e,I}^*$  from part (1) has  $\leq |I| + \aleph_0$  equivalence classes.
- 2) If  $J \in K_{\tau_{\alpha(*)}}^{\text{lin}}$  and  $e$  as in part (2),  $\theta = \text{cf}(J)$  and  $|J| < \lambda = \lambda^{<\theta} < \lambda^\theta$  then there is  $I \in K_{\tau_{\alpha(*)}}^{\text{lin}}$  of cardinality  $\lambda$  such that  $\mathcal{E}_{e,I}^*$  has  $\lambda^\theta$  equivalence classes.

*Remark.* We can define the stability spectrum for some classes, essentially this is done in §7, generally we intend to look at it in [Sh:F782].

## §7 CATEGORICITY FOR A.E.C. WITH BOUNDED AMALGAMATION

Recall that 4.10 is the main result of this chapter; we think that it will lead to understanding the categoricity spectrum of an a.e.c. In particular we hope eventually to prove that this spectrum contains or is disjoint to some end segments of the class of cardinals. Still here we like to show that what we have is enough at least for restricted enough families of a.e.c.  $\mathfrak{K}$ 's, those definable by  $\mathbb{L}_{\kappa,\omega,\kappa}$  a measurable cardinal or with enough amalgamation (concerning them and earlier results see Chapter N). We could have relied on<sup>8</sup> [Sh 394], but though we mention connections, we do not rely on it, preferring self-containment.

We can say much even if we replace categoricity by strong solvability, but do this only when it is cheap; we can work even with weak and even pseudo-solvability but not here.

**7.1 Hypothesis.** 1)  $\mathfrak{K}$  is an a.e.c., so  $\mathcal{S}(M) = \mathcal{S}_{\mathfrak{K}_\lambda}(M)$  for  $M \in K_\lambda$ , see III.

 600-0.12

2) Let  $K_\mu^x$  be the class  $K_\mu$  if  $K$  is categorical in  $\mu$  and the class of superlimit models in  $\mathfrak{K}_\mu$  if there is one, (the two definitions are compatible).

The following is a crucial claim because lack of locality is the problem in [Sh 394].

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<sup>8</sup>In the references to [Sh 394], e.g. 1.6tex is to 1.6 in the published version and 1.8 is in the e-version.

**7.2 Claim.** *Assume*

- (a)  $\text{cf}(\mu) > \kappa \geq \text{LS}(\mathfrak{K})$
- (b)  $\mathfrak{K}_{<\mu}$  has amalgamation
- (c)  $\Phi \in \Upsilon_{\kappa}^{\text{or}}[\mathfrak{K}]$  satisfies: if  $I$  is  $\theta$ -wide and  $\theta \in (\kappa, \mu)$  then  $\text{EM}_{\tau(\mathfrak{K})}(I, M)$  is  $\theta$ -saturated (see 0.14(1), III. 600-0.15 (2) and III. 600-0.19).

Then

- ( $\alpha$ ) for some  $\mu_* < \mu$ , the class  $\{M \in K_{<\mu} : M \text{ is saturated}\}$  is  $[\mu_*, \mu)$ -local, see Definition 7.4(3) below
- ( $\alpha$ )<sup>+</sup> this applies not only to  $\mathcal{S}(M) = \mathcal{S}^1(M)$  but also for  $\mathcal{S}^\partial(M)$  if  $\text{cf}(\mu) > \kappa^\partial$ .

Recall

**7.3 Definition.**  $\mathfrak{K}$  is  $\mu$ -stable if  $\mu \geq \text{LS}(\mathfrak{K})$  and  $M \in K_{\leq \mu} \Rightarrow |\mathcal{S}(M)| \leq \mu$ .

Recall ([Sh 394, Def.1.8=1.6tex](1),(2)).

**7.4 Definition.** 1) For  $M \in \mathfrak{K}$ ,  $\mu \geq \text{LS}(\mathfrak{K})$ , satisfying  $\mu \leq \|M\|$  and  $\alpha$ , let  $\mathbb{E}_{M, \mu, \alpha}$  be the following equivalence relation on  $\mathcal{S}^\alpha(M)$ :  $p_1 \mathbb{E}_{M, \mu, \alpha} p_2$  iff for every  $N \leq_{\mathfrak{K}} M$  of cardinality  $\mu$  we have  $p_1 \restriction N = p_2 \restriction N$ . We may suppress  $\alpha$  if it is 1, similarly below; let  $\mathbb{E}_{\mu, \alpha}$  be  $\bigcup \{\mathbb{E}_{M, \mu, \alpha} : M \in K\}$  and so  $\mathbb{E}_\mu = \mathbb{E}_{\mu, 1}$ .

2) We say that  $M \in \mathfrak{K}$  is  $\mu - \alpha$ -local when  $\mathbb{E}_{M, \mu, \alpha}$  is the equality; we say that  $p \in \mathcal{S}^\alpha(M)$  is  $\mu$ -local if  $p/\mathbb{E}_{M, \mu, \alpha}$  is a singleton and we say e.g.  $K' \subseteq \mathfrak{K}$  is  $\mu - \alpha$ -local (in  $\mathfrak{K}$ , if not clear from the context) when every  $M \in K'$  is.

3) We say  $K' \subseteq \mathfrak{K}$  is  $[\mu_*, \mu) - \alpha$ -local if every  $M \in K' \cap \mathfrak{K}_{[\mu_*, \mu)}$  is  $\mu_* - \alpha$ -local.

4) We say that  $\bar{a} \in N$  realizes  $\mathbf{p} \in \mathcal{S}_{\mathfrak{K}}^\alpha(M)/\mathbb{E}_{\mu, \alpha}$  if  $M \leq_{\mathfrak{K}} N$  and for every  $M' \leq_{\mathfrak{K}} N$  of cardinality  $\mu$  the sequence  $\bar{a}$  realizes  $\mathbf{p} \restriction M'$  in  $N$  or pedantically realizes  $q \restriction M'$  for some, equivalently every  $q \in \mathbf{p}$ .

*Remark.* If  $M \in \mathfrak{K}_\mu$ , then  $M$  is  $\mu - \alpha$ -local.

*Proof of 7.2.* Recall  $\Phi \in \Upsilon_{\kappa}^{\text{or}}[\mathfrak{K}]$ , see Definition 0.8(2) and Claim 0.9. Easily there is  $\langle I_\theta : \theta \in [\kappa, \mu) \rangle$ , an increasing sequence of wide linear orders which are strongly  $\aleph_0$ -homogeneous (that is dense with neither first nor last element such that if  $n < \omega$  and  $\bar{s}, \bar{t} \in {}^n(I_\theta)$  are  $<_I$ -increasing then some automorphism of  $I_\theta$  maps  $\bar{s}$  to  $\bar{t}$ , e.g. the order of any real closed field/or just ordered field) satisfying  $|I_\theta| = \theta$ .

Recalling  $\mathbb{Q}$  here is the rational order, we let  $J_\theta = \mathbb{Q} + I_\theta$ ,  $M_\theta = \text{EM}_{\tau(\mathfrak{K})}(I_\theta, \Phi)$  and  $N_\theta = \text{EM}_{\tau(\mathfrak{K})}(J_\theta, \Phi)$ . So

- ⊗ (a)  $M_\theta \leq_{\mathfrak{K}_\theta} N_\theta$
- (b)  $M_{\theta_1} \leq_{\mathfrak{K}} M_{\theta_2}$  and  $N_{\theta_1} \leq_{\mathfrak{K}} N_{\theta_2}$  when  $\kappa \leq \theta_1 < \theta_2 < \mu$
- (c)  $M_\theta$  is saturated (for  $\mathfrak{K}$ , of course) when  $\theta > \kappa$
- (d) every type from  $\mathcal{S}(M_\theta)$  is realized in  $N_\theta$
- (e) if  $n < \omega$ ,  $\bar{a} \in {}^n(N_\theta)$  then for some  $\bar{a}' \in {}^n(N_\kappa)$  and automorphism  $\pi$  of  $N_\theta$ ,  $\pi(\bar{a}) = \bar{a}'$  and  $\pi$  maps  $M_\theta$  onto itself.

[Why? Clauses (a),(b) holds by clause (c) of Claim 0.9(1) recalling Definition 0.8(2).

Clause (c) holds by Clause (c) of the assumption of 7.2; you may note [Sh 394, 6.7=6.4tex](2).

Clause (d) holds as  $\text{EM}_{\tau(\mathfrak{K})}(\theta^+ + J_\theta, \Phi) \in \mathfrak{K}_{\theta^+}$  is saturated, and use the definition of a type (or like the proof of clause (e) below using appropriate  $I' + I_\theta$  instead  $\theta^+ + J_\theta$ ); you may note [Sh 394, 6.8=6.5tex].

Clause (e) holds as for every finite sequence  $\bar{t}$  from  $J_\theta$  there is an automorphism  $\pi$  of  $J_\theta$  such that:  $\pi$  is the identity on  $\mathbb{Q}$ , it maps  $I_\theta$  onto itself and it maps  $\bar{t}$  to a sequence from  $J_\kappa = \mathbb{Q} + I_\kappa$ , such  $\pi$  exists as  $I_\theta$  is strongly  $\aleph_0$ -homogeneous and  $I_\kappa \subseteq I_\theta$  is infinite.]

For any  $a \neq b$  from  $N_\kappa$  let

$$\mu(a, b) = \text{Min}\{\theta : \theta \geq \kappa \text{ and if } \theta < \mu \\ \text{then } \mathbf{tp}_{\mathfrak{K}}(a, M_\theta, N_\theta) \neq \mathbf{tp}_{\mathfrak{K}}(b, M_\theta, N_\theta)\}.$$

So  $\mu(a, b) \leq \mu$ . Let

$$\mu_* = \sup\{\mu(a, b) : a, b \in N_\kappa \text{ and } \mu(a, b) < \mu\}.$$

So  $\mu_*$  is defined as the supremum on a set of  $\leq \kappa \times \kappa$  cardinals  $< \mu$ , which is a cardinal of cofinality  $\text{cf}(\mu) > \kappa$ , hence clearly  $\mu_* < \mu$ . Also  $\mu_* \geq \kappa$  as there are  $a \neq b$  from  $M_\kappa$  hence  $\mu(a, b) = \kappa$ . Now suppose that  $\theta \in [\mu_*, \mu)$ ,  $M \in \mathfrak{K}_\theta$  is saturated and  $p_1 \neq p_2 \in \mathcal{S}(M)$  and we shall find  $M' \leq_{\mathfrak{K}} M$ ,  $M' \in \mathfrak{K}_{\mu_*}$  such that  $p_1 \restriction M' \neq p_2 \restriction M'$ , this suffice.

Clearly  $M_\theta \in K_\theta$  is saturated (by clause (c) of ⊗) hence the models  $M, M_\theta$  are isomorphic so without loss of generality  $M = M_\theta$ . But by clause (d) of ⊗ every type from  $\mathcal{S}(M_\theta)$  is realized in  $N_\theta$ , so let  $b_\ell$  be such that  $p_\ell = \mathbf{tp}_{\mathfrak{K}}(b_\ell, M_\theta, N_\theta)$  for  $\ell = 1, 2$ . Now there is an automorphism  $\pi$  of  $N_\theta$  which maps  $M_\theta$  onto itself

and maps  $b_1, b_2$  into  $N_\kappa$  (by clause (e) of  $\otimes$ ) and let  $a_\ell = \pi(b_\ell)$  for  $\ell = 1, 2$ , so  $a_1, a_2 \in N_\kappa$ .

Now

$$\begin{aligned} \mathbf{tp}(a_1, M_\theta, N_\theta) &= \mathbf{tp}(\pi(b_1), \pi(M_\theta), \pi(N_\theta)) = \pi(\mathbf{tp}(b_1, M_\theta, N_\theta)) \neq \\ &\neq \pi(\mathbf{tp}(b_2, M_\theta, N_\theta)) = \mathbf{tp}(\pi(b_2), \pi(M_\theta), \pi(N_\theta)) = \mathbf{tp}(a_2, M_\theta, N_\theta). \end{aligned}$$



Hence by the definition of  $\mu(a_1, a_2)$  we have  $\mu(a_1, a_2) \leq \theta < \mu$ . Hence by the definition of  $\mu_*$  we have  $\mu(a_1, a_2) \leq \mu_*$  which implies that  $\mathbf{tp}_{\mathfrak{K}}(a_1, M_{\mu_*}, N_{\mu_*}) \neq \mathbf{tp}_{\mathfrak{K}}(a_2, M_{\mu_*}, N_{\mu_*})$ . As  $\pi$  is an automorphism of  $N_\theta$  and  $M_{\mu_*} \leq_{\mathfrak{K}} M_\theta$  it follows that  $\mathbf{tp}_{\mathfrak{K}}(\pi^{-1}(a_1), \pi^{-1}(M_{\mu_*}), \pi^{-1}(N_\theta)) \neq \mathbf{tp}_{\mathfrak{K}}(\pi^{-1}(a_2), \pi^{-1}(M_{\mu_*}), \pi^{-1}(N_\theta))$  which means

$\mathbf{tp}_{\mathfrak{K}}(b_1, \pi^{-1}(M_{\mu_*}), N_\theta) \neq \mathbf{tp}_{\mathfrak{K}}(b_2, \pi^{-1}(M_{\mu_*}), N_\theta)$ , but  $\pi^{-1}(M_{\mu_*}) \leq_{\mathfrak{K}} M_\theta$  as  $\pi$  maps  $M_\theta$  onto itself and recall that  $p_\ell = \mathbf{tp}_{\mathfrak{K}}(b_\ell, M_\theta, N_\theta)$  so  $p_\ell \upharpoonright \pi^{-1}(M_{\mu_*})$  is well defined for  $\ell = 1, 2$ . Hence  $p_1 \upharpoonright \pi^{-1}(M_{\mu_*}) \neq p_2 \upharpoonright \pi^{-1}(M_{\mu_*})$  and clearly  $\pi^{-1}(M_{\mu_*})$  has cardinality  $\mu^*$  and is  $\leq_{\mathfrak{K}} M_\theta$ , so we are done proving clause  $(\alpha)$ . The proof of clause  $(\alpha)^+$  is the same except that

- (\*)<sub>1</sub> if  $\theta \in [\kappa, \mu), \bar{t} \in {}^\partial(I_\theta)$  then some automorphism  $\pi$  of  $I_\theta$  maps  $\bar{t}$  to some  $\bar{t}' \in {}^\partial(I_\kappa)$ , justified by 5.1
- (\*)<sub>2</sub> we replace  $\mathbb{Q}$  by  $\partial^+$
- (\*)<sub>3</sub>  ${}^\partial(N_\kappa)$  has cardinality  $\leq (\partial^+ + \kappa)^\partial \leq \kappa^\partial < \text{cf}(\mu)$ .

□<sub>7.2</sub>

Implicit in non- $\mu$ -splitting is

**7.5 Definition.** Assume  $\alpha < \mu^+$ ,  $N \in K_{\leq \mu}$ ,  $N \leq_{\mathfrak{K}} M$  and  $p \in \mathcal{S}^\alpha(M)$  does not  $\mu$ -split over  $N$ , see Definition IV.  705-gr.1  (1). The scheme of the non- $\mu$ -splitting,  $\mathbf{p} = \text{sch}_\mu(p, N)$  is  $\{(N'', c, \bar{b})_{c \in N} / \cong$ : we have  $N \leq_{\mathfrak{K}} N' \leq_{\mathfrak{K}} M$  and  $N' \leq_{\mathfrak{K}} N''$ ,  $\{N', N''\} \subseteq K_\mu$  and the sequence  $\bar{b}$  realizes  $p \upharpoonright N'$  in the model  $N''\}$ .

**7.6 Definition.** For a cardinal  $\mu$  and model  $M$  let

1)

$$\begin{aligned} \text{ps} - \mathcal{S}_\mu(M) &= \mathcal{S}_{\mathfrak{K}, \mu}(M) = \{\mathbf{p} : \mathbf{p} \text{ is a function with domain } \{N \in K_\mu : N \leq_{\mathfrak{K}} M\} \\ &\text{such that } \mathbf{p}(N) \in \mathcal{S}(N) \text{ and } N_1 \leq_{\mathfrak{K}} N_2 \in \text{Dom}(\mathbf{p}) \\ &\Rightarrow \mathbf{p}(N_1) = \mathbf{p}(N_2) \upharpoonright N_1\}. \end{aligned}$$

2) For  $p \in \mathcal{S}(M)$  let  $p \upharpoonright (\leq \mu)$  be the function  $\mathbf{p}$  with domain  $\{N \in K_\mu : N \leq_{\mathfrak{K}} M\}$  such that  $\mathbf{p}(N) = p \upharpoonright N$ .

**7.7 Observation.** 1) The function  $p \mapsto p \upharpoonright (\leq \mu)$  is a function from  $\mathcal{S}(M)$  into  $\text{ps-}\mathcal{S}_\mu(M)$  such that for  $p_1, p_2 \in \mathcal{S}(M)$  we have  $p_1 \upharpoonright (\leq \mu) = p_2 \upharpoonright (\leq \mu) \Leftrightarrow p_1 \mathbb{E}_\mu p_2$ .  
 2) The subset  $\{p \upharpoonright (\leq \mu) : p \in \mathcal{S}(M)\}$  of  $\text{ps-}\mathcal{S}_\mu(M)$  has cardinality  $|\mathcal{S}(M)/\mathbb{E}_\mu|$ .

*Proof.* Should be clear. □<sub>7.7</sub>

**7.8 Claim.** *Every (equivalently some)  $M \in K_\mu^x$  is  $\lambda^+$ -saturated when:*

- (a) (α)  $\mathfrak{K}$  is categorical in  $\mu$   
       or just  
       (β)  $\mathfrak{K}$  is strongly solvable in  $\mu$
- (b)  $\text{LS}(\mathfrak{K}) \leq \lambda < \chi \leq \mu$  and  $2^{2^\lambda} \leq \mu$  (actually  $2^\lambda \leq \mu$  suffice)
- (c) (α)  $\aleph_{\lambda+4} = \lambda^{+\lambda+4} \leq \chi$   
       or at least  
       (β) if  $\theta = \text{cf}(\theta) \leq \lambda$  is  $\aleph_0$  or a measurable cardinal then for some  
            $\partial \in (\lambda, \chi)$  we have:  $\partial = \partial^{<\theta} < \partial^\theta$  or at least  $\partial^{<\theta^{\text{tr}}} > \partial$  (i.e. there  
           is a tree  $\mathcal{T}$  with  $\theta$  levels,  $\partial$  nodes and the number of  $\theta$ -branches  
           of  $\mathcal{T}$  is  $> \chi$ , see [Sh 589])
- (d)  $\mathfrak{K}_{\geq \partial} \neq \emptyset$  for every  $\partial$ , equivalently  $K_{\geq \partial} \neq \emptyset$  for arbitrarily large  $\theta < \beth_{1,1}(\text{LS}(\mathfrak{K}))$
- (e) (α)  $\mathfrak{K}_{<\mu}$  has amalgamation and JEP  
       or just  
       (β) if  $\text{LS}(\mathfrak{K}) \leq \partial < \chi$  then
  - (i)  $\mathfrak{K}_\partial$  has amalgamation and JEP and
  - (ii)  $\mathfrak{K}$  has  $(\partial, \leq \partial^+, \mu)$ -amalgamation<sup>9</sup> (see I. 88r-2.5 (2))  
       hence<sup>10</sup>
  - (iii) every  $M \in K_{\partial^+}$  has a  $\leq_{\mathfrak{K}}$ -extension in  $K_\mu^x$   
       (actually (i) + (iii) suffices).

*Remark.* 1)  $M$  is  $\lambda^+$ -saturated is well defined as  $\mathfrak{K}_{\leq \lambda}$  has amalgamation.

2) We assume  $2^{2^\lambda} \leq \mu$  because the proof is simpler with not much loss (at least as

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<sup>9</sup>It suffices to have: if  $M_0 \leq_{\mathfrak{K}} M_1 \in K_{\partial^+}$ ,  $M_1 \leq_{\mathfrak{K}} M_2 \in K_\mu^x$  and  $M_0 \in K_\partial$  then  $M_1$  can be  $\leq_{\mathfrak{K}}$ -embedded into some  $M_3 \in K_\mu^x$ . Similarly in 7.12.

<sup>10</sup>Why? Assume  $M \in K_{\partial^+}$  let  $M_2 \in K_\mu^x$ , let  $M_0 \leq_{\mathfrak{K}} M_2$  be of cardinality  $\partial$ , let  $M_1 \in K_{\partial^+}$  be a  $\leq_{\mathfrak{K}}$ -extension of  $M_0$  which there is an  $\leq_{\mathfrak{K}}$ -embedding  $f$  of  $M$  into  $M_1$  (exists as  $\mathfrak{K}_\partial$  has amalgamation and JEP). Lastly, use “ $\mathfrak{K}$  has  $(\partial, \leq \partial^+, \mu)$ -amalgamation

long as other parts of the analysis are not much tighter).

3) We can weaken the assumptions. In particular using solvability instead categoricity, but for non-essential reasons this is delayed; similarly in 7.12.

4) If  $\mu = \mu^\lambda$  the claim is easy (as in §1).

*Proof.* Note that by [Sh:g, IX,§2], [Sh:g, II,3.1] if clause (c)( $\alpha$ ) holds then clause (c)( $\beta$ ) holds, hence we can assume (c)( $\beta$ ).

Let  $\Phi \in \Upsilon_{\mathfrak{K}}^{\text{or}}$  see Definition 0.8(2), exist by 0.9 and clause (d) of the assumption and  $I \in K_\mu^{\text{lin}} \Rightarrow \text{EM}_{\tau(\mathfrak{K})}(I, \Phi) \in K_\mu^x$  (trivially if  $K$  is categorical in  $\mu$ , otherwise by the definition of solvable).

Clearly

(\*)<sub>0</sub> if  $\partial \in [\text{LS}(\mathfrak{K}), \chi)$  then  $\mathfrak{K}$  is stable in  $\partial$ .

[Why? We prove assuming clause (e)( $\beta$ ), as the case of clause (e)( $\alpha$ ) is easier. Otherwise as  $\mathfrak{K}_\partial$  has amalgamation there are  $M_0 \leq_{\mathfrak{K}} M_1$  such that  $M_0 \in K_\partial$ ,  $M_1 \in K_{\partial^+}$  and  $\{\text{tp}_{\mathfrak{K}}(a, M_0, M_1) : a \in M_1\}$  has cardinality  $\partial^+$ . By assumption (e)( $\beta$ )(iii) there is  $N_1$  such that  $M_1 \leq_{\mathfrak{K}} N_1 \in \mathfrak{K}_\mu$  and without loss of generality  $N_1 \in K_\mu^x$ . Let  $I$  be as in 5.1 with  $(\lambda, \theta_2, \theta_1, \mu)$  there standing for  $(\mu, \partial^{++}, \partial^+, \partial)$  here and  $N_2 := \text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ . Now by 5.1(2),  $N_1 \not\cong N_2$ , contradiction to “ $\mathfrak{K}$  categorical in  $\mu$ ”. Or you may see [Sh 394, 1.7=1.5tex].]



The proof now splits to two cases.

Case 1: For every  $M \in K_\mu^x$  we have  $\mu \geq |\mathcal{S}(M)/\mathbb{E}_\lambda|$ .

For every  $M \in K_\mu^x$  there is  $M'$  such that:  $M \leq_{\mathfrak{K}} M' \in K_\mu$  and for every  $\mathbf{p} \in \mathcal{S}(M)/\mathbb{E}_\lambda$  either  $\mathbf{p}$  is realized in  $M'$  or there are no  $M'', a$  such that  $M' \leq_{\mathfrak{K}} M'' \in K_\mu$  and  $a \in M''$  realizes  $\mathbf{p}$  in  $M''$ .

[Why? Let  $\langle p_i/\mathbb{E}_\lambda : i < \mu \rangle$  list  $\mathcal{S}(M)/\mathbb{E}_\lambda$ , exists by the assumptions and choose  $M_i$  for  $i \leq \mu$ ,  $\leq_{\mathfrak{K}_\mu}$ -increasing continuous such that  $M_{i+1}$  satisfies the demand for  $\mathbf{p} = p_i/\mathbb{E}_\lambda$ , possibly no  $p \in p_i/\mathbb{E}_\lambda$  has an extension in  $\mathcal{S}(M_{i+1})$  (hence is not realized in it), so then the desired demand holds trivially; note that it is not unreasonable to assume  $\mathfrak{K}_\mu$  has amalgamation and it clarifies but it is not necessary.]

Also without loss of generality  $M' \in K_\mu^x$  as any model  $M$  from  $K_\mu$  has a  $\leq_{\mathfrak{K}}$ -extension in  $K_\mu^x$  (at least if  $M$  does  $\leq_{\mathfrak{K}}$ -extend some  $M' \in K_\mu^x$ ).

Now we can choose by induction on  $i \leq \lambda^+$  a model  $M_i \in K_\mu^x$ ,  $\leq_{\mathfrak{K}}$ -increasing continuous with  $i$ , such that for every  $p \in \mathcal{S}(M_i)$  either there is  $q \in \mathcal{S}(M_i)$  realized in  $M_{i+1}$  which is  $\mathbb{E}_\lambda$ -equivalent to  $p$  or there is no  $\leq_{\mathfrak{K}}$ -extension of  $M_{i+1}$  satisfying this. Now we shall prove that  $M_{\lambda^+}$  is  $\lambda^+$ -saturated recalling Definition III.  600-0.15 . Now if  $N \leq_{\mathfrak{K}} M_{\lambda^+}$ ,  $\|N\| \leq \lambda$  and  $p \in \mathcal{S}(N)$  then there is  $i < \lambda^+$  such that  $N \leq_{\mathfrak{K}} M_i$  and we can find  $p' \in \mathcal{S}(M_{\lambda^+})$  extending

$p$ . (Why? If clause (e)( $\alpha$ ) holds then this follows by  $\mathfrak{K}_{<\mu}$  having amalgamation, see I. 88r-2.8. If clause (e)( $\beta$ ) holds, use “ $\mathfrak{K}$  has the  $(\lambda, \leq \lambda^+, \mu)$ -amalgamation property” recalling  $\text{LS}(\mathfrak{K}) \leq \lambda < \chi$ .) Hence there is  $a \in M_{i+1}$  such that  $\text{tp}(a, M_i, M_{i+1})\mathbb{E}_\lambda(p' \upharpoonright M_i)$ , hence  $a$  realizes  $p$  in  $M_{i+1}$  hence in  $M_{\lambda^+}$ .

Case 2: Not Case 1.

Let  $I$  be as in 5.1 with  $(\lambda, \theta_2, \theta_1, \mu)$  there standing for  $(\mu, \lambda^{++}, \lambda^+, \lambda)$  here, so  $|I| = \mu$ . Let  $M = \text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ , so by not Case 1 we can find  $p_i \in \mathcal{S}(M)$  for  $i < \mu^+$  pairwise non- $\mathbb{E}_\lambda$ -equivalent. As  $\mathfrak{K}_\lambda$  is a  $\lambda$ -a.e.c. with amalgamation and is stable in  $\lambda$  (by  $(*)_0$ ) we can deduce, see IV. 705-gr.6 (2), that: if  $p \in \mathcal{S}(M)$  then for some  $N \leq_{\mathfrak{K}} M$  of cardinality  $\lambda$  the type  $p$  does not  $\lambda$ -split over  $N$  (or see [Sh 394, 3.2 = 3.2tex](1)). For each  $i$  choose  $N_i \leq_{\mathfrak{K}} M$  of cardinality  $\lambda$  such that  $p_i$  does not  $\mu$ -split over  $N_i$ . As there is no loss in increasing  $N_i$  (as long as it is  $\leq_{\mathfrak{K}} M$  and has cardinality  $\lambda$ ) without loss of generality

(\*)<sub>1</sub>  $N_i = \text{EM}_{\tau(\mathfrak{K})}(I_i, \Phi)$  where  $I_i \subseteq I$  and  $|I_i| = \lambda$  and let  $\bar{t}_i = \langle t_\varepsilon^i : \varepsilon < \lambda \rangle$  list  $I_i$  with no repetitions.

As  $2^\lambda \leq \mu$  without loss of generality the  $I_i$ 's are pairwise isomorphic, so without loss of generality for  $i, j < \mu^+$ , the mapping  $t_\varepsilon^i \mapsto t_\varepsilon^j$  is such an isomorphism. Moreover, without loss of generality

(\*)<sub>2</sub> for every  $i, j < \mu^+$  there is an automorphism  $\pi_{i,j}$  of  $I$  mapping  $t_\varepsilon^i$  to  $t_\varepsilon^j$  for  $\varepsilon < \lambda$ .

[Why? By 5.1(1) as we can replace  $\langle p_i : i < \mu^+ \rangle$  by  $\langle p_i : i \in \mathcal{U} \rangle$  for every unbounded  $\mathcal{U} \subseteq \mu^+$ .]

Let  $\mathfrak{p}_i$  be the non- $\lambda$ -splitting scheme of  $p$  over  $N_i$  (see Definition 7.5). Without loss of generality:

(\*)<sub>3</sub> for  $i, j < \mu^+$ , the isomorphism  $h_{i,j}$  from  $N_j = \text{EM}_{\tau(\mathfrak{K})}(I_j, \Phi)$  onto  $N_i = \text{EM}_{\tau(\mathfrak{K})}(I_i, \Phi)$  induced by the mapping  $t_\zeta^j \mapsto t_\zeta^i$  (for  $\zeta < \lambda$ ) satisfies

(i) it is an isomorphism from  $N_j$  onto  $N_i$

(ii) it maps  $\mathfrak{p}_j$  to  $\mathfrak{p}_i$ .

[Why? For (i) this holds by the definition of  $\text{EM}(I_i, \Phi)$ . For (ii) let  $h_{i,0}$  map  $\mathfrak{p}_i$  to  $\mathfrak{p}'_i$ . The number of schemes is  $\leq 2^{2^\lambda}$ ; so if  $\mu \geq 2^{2^\lambda}$  then without loss of generality  $i < \mu^+ \Rightarrow \mathfrak{p}'_i = \mathfrak{p}'_1$  hence we are done (with no real loss). If we weaken the assumption  $\mu \geq 2^{2^\lambda}$  to  $\mu \geq 2^\lambda$  (or even  $\mu > \lambda$  so waive  $(*)_2$ ) using 5.1(4) we can find  $I_i^+$  such that  $I_i \subseteq I_i^+ \subseteq I$ ,  $|I_i^+| \leq \lambda^+$  and for every  $J \subseteq I$  of cardinality  $\leq \lambda$  there is an automorphism of  $I$  over  $I_i$  mapping  $J$  into  $I_i^+$ . So only

$\langle \mathfrak{p}'_i((\text{EM}_{\tau(\mathfrak{K})}(I_0^+, \Phi), c, \bar{b}))_{c \in \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)} / \cong : \bar{b} \in {}^\lambda(\text{EM}_{\tau(\mathfrak{K})}(I_0^+, \Phi)) \rangle$  matters (an overkill) but this is determined by  $p_i \upharpoonright \text{EM}_{\tau(\mathfrak{K})}(I_0^+, \Phi)$  which  $\in \mathcal{S}(\text{EM}_{\tau(\mathfrak{K})}(I_0^+, \Phi))$  by  $(*)_0$  and as  $\mathfrak{K}$  is stable in  $\lambda^+$  without loss of generality  $\mathfrak{p}'_{1+i} = \mathfrak{p}'_1$  and we are done.]

Now we translate our problem to one on expanded (by unary predicates) linear orders which was treated in §6. Recall that by 5.1(3), we can use  $I = \text{EM}_{\{<\}}(I^*, \Psi)$  where  $\Psi \in \Upsilon_{\aleph_0}^{\text{lin}}[2]$ , see Definition 0.11(5), and  $I^* = I_{\lambda, \mu \times \lambda^+}^{\text{lin}}$  from 6.12(2) with  $\alpha(*) = 2$ . Recall that  $I^* = I_{\lambda, \mu \times \lambda^{++}}^{\text{lin}}$  is  $\mu \times \lambda^{++}$  expanded by  $P_1 = \{\alpha \in I^* : \text{cf}(\alpha) \geq \lambda^+\}$ ,  $P_0 = I^* \setminus P_1$  so  $I^*$  is a well ordered  $\tau_2^*$ -model, i.e.  $\in K_{\tau_2^*}^{\text{lin}}$ , see Definition 0.11(5). Without loss of generality  $I_i = \text{EM}_{\{<\}}(I_i^*, \Psi)$  where  $I_i^* \subseteq I^*$  has cardinality  $\lambda$  and the pair  $(I^*, I_i^*)$  is a reasonable  $(\lambda, \alpha(*))$ -base which is a wide  $(\mu, \lambda, \alpha(*))$ -base, see Definition 6.10(3)(4), Claim 6.12(2). Without loss of generality for every  $i < \mu^+$  there is  $h_i$ , an isomorphism from  $I_0^*$  onto  $I_i^*$  such that (see below) the induced function  $h_1^{[1]}$  maps  $\bar{t}_0$  to  $\bar{t}_i$ . Let  $J^* = I_0^*$  and  $J = I_0$ . We like to apply §6 for  $J^*, I^*$  fixing  $\alpha(*) = 2$ ,  $\bar{u}^* = (u^-, u^+) = (\{0\}, \emptyset)$ . So recalling Definition 6.3(2) for every  $h \in \text{inc}_{J^*}^{\bar{u}^*}(I^*)$  we can naturally define the function  $h^{[1]}$  by  $h^{[1]}(\sigma^{\text{EM}(J^*, \Psi)}(t_0, \dots, t_{n-1})) = \sigma^{\text{EM}(J^*, \Psi)}(a_{h(t_0)}, \dots, a_{h(t_{n-1})})$  whenever  $\sigma(x_0, \dots, x_{n-1})$  is a  $\tau(\Psi)$ -term and  $J^* \models "t_0 < \dots < t_{n-1}"$  so it is an isomorphism from  $\text{EM}_{\{<\}}(J^*, \Psi)$  onto  $\text{EM}_{\{<\}}(I^* \upharpoonright \text{Rang}(h), \Psi)$  so as  $J^* \subseteq I^*$  by 5.1(5) there is an automorphism  $h^{[2]}$  of  $I$  extending  $h^{[1]}$  and so there is an automorphism  $h^{[3]}$  of  $\text{EM}(I, \Phi)$  such that  $h^{[3]}(a_t) = a_{h^{[2]}(t)}$  for  $t \in I$  and  $h^{[3]}(\sigma^{\text{EM}(I, \Phi)}(a_{t_0}, \dots, a_{t_{n-1}})) = \sigma^{\text{EM}(I, \Phi)}(a_{h^{[2]}(t_0)}, \dots, a_{h^{[2]}(t_{n-1})})$  where  $t_0 <_I \dots <_I t_{n-1}$  and  $\sigma(x_0, \dots, x_{n-1})$  is a  $\tau(\Phi)$ -term.

Note that

$(*)_4$  if  $h', h''$  are automorphisms of  $\text{EM}_{\tau[\mathfrak{K}]}(I, \Phi)$  extending  $h^{[3]} \upharpoonright \text{EM}_{\tau[\mathfrak{K}]}(I_0)$  then  $h'(p_0/\mathbb{E}_\lambda) = h''(p_0/\mathbb{E}_\lambda)$ .

[Why? Because  $p_0$  does not  $\lambda$ -split over  $\text{EM}_{\tau[\mathfrak{K}]}(I_0, \Phi)$ .]

We define a two-place relation  $\mathcal{E}$  on  $\text{inc}_{J^*}(I^*)$  by:  $h_1 \mathcal{E} h_2$  if  $h_1^{[3]}(p_0/\mathbb{E}_\lambda) = h_2^{[3]}(p_0/\mathbb{E}_\lambda)$ . (Note that  $h \mapsto h^{[3]}$  is a function so this is well defined and  $h^{[3]}$  is an automorphism of  $\text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ ). By  $(*)_4$  clearly  $\mathcal{E}$  is an invariant equivalence relation on  $\text{inc}_{J^*}^{\bar{u}^*}(I^*)$  with  $> \mu$  equivalence classes as exemplified by  $\langle h_i : i < \mu^+ \rangle$ .

By 6.19 there is  $e \in \mathbf{e}(J^*, I^*)$  such that (recalling Definition 6.16) the filter  $\mathcal{D}_{\mathcal{E}, e}$  has an extension to a non-principal ultrafilter  $\mathcal{D}$  so for some regular  $\theta \leq \lambda$  there is a function  $g$  from  $\text{Dom}(\mathbf{e})/e$  onto  $\theta$  which maps  $\mathcal{D}$  to a uniform ultrafilter  $g(\mathcal{D})$  on  $\theta$ , so  $\partial^{<\theta>_{\text{tr}}} \leq \partial^{\text{Dom}(\mathbf{e})/e}/\mathcal{D}_{\mathcal{E}, e}$  for every cardinal  $\partial$ . Choose such a pair  $(g, \theta)$  with minimal  $\theta$  so  $\mathcal{D}$  is  $\theta$ -complete hence  $\theta = \aleph_0$  or  $\theta$  is a measurable cardinal  $\leq \lambda$ . By clause (c)( $\beta$ ) of our assumption justified in the beginning of the proof there is  $\partial \in (\lambda^+, \chi)$  such that  $\partial < \partial^{<\theta>_{\text{tr}}}$  hence  $\partial^+ \leq \partial^{<\theta>_{\text{tr}}} \leq \partial^{\text{Dom}(\mathbf{e})/e}/\mathcal{D}_{\mathcal{E}, e}$ .



So letting  $I_\partial^0 = I_{\lambda, \partial \times \lambda^{++}}^{\text{lin}} \subseteq I^*$  the set  $\{\bar{t}/\mathcal{E} : \bar{t} \in \text{incr}_{J^*}(I^*) \text{ and } \text{Rang}(\bar{t}) \subseteq I_\partial^0\}$  has cardinality  $> \partial$ . Now for each  $\bar{t} \in \text{inc}_{J^*}^{\bar{u}^*}(I^*)$  let  $\pi_{\bar{t}} \in \text{Aut}(I)$  be such that  $\pi_{\bar{t}}(\bar{t}_0) = \bar{t}$  and let  $\hat{\pi}_{\bar{t}}$  be the automorphism of  $\text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$  which  $\pi_{\bar{t}}$  induce, and let  $p_t = \hat{\pi}_{\bar{t}}(p_0) \in \mathcal{S}(M)$ . Hence  $\{\hat{\pi}_{\bar{t}}(p_0) \upharpoonright \text{EM}_{\tau(\mathfrak{K})}(I_{\lambda, \partial \times \lambda^{++}}^{\text{lin}}, \Phi) : \bar{t} \in \text{inc}_{J^*}^{\bar{u}^*}(I^*) \text{ and } \text{Rang}(\bar{t}) \subseteq I_{\lambda, \partial \times \lambda^{++}}^{\text{lin}}\}$  is of cardinality  $> \partial$ , contradicting “ $\mathfrak{K}$  stable in  $\partial$ ” from  $(*)_0$ .  $\square_{7.8}$

Note but we shall not use

**7.9 Conclusion.** 1) Under the assumptions of 7.8 we have  $\kappa(\mathfrak{K}_\mu) = \aleph_0$ , see below.  
2) Moreover,  $\kappa_{\text{st}}(\mathfrak{K}_\mu) = \emptyset$ .

Recall

**7.10 Definition.** If  $\mathfrak{K}_\mu$  is an  $\mu$ -a.e.c. with amalgamation which is stable, then:

- (a)  $\kappa(\mathfrak{K}_\mu) = \aleph_0 + \sup\{\kappa^+ : \kappa \text{ regular} \leq \mu \text{ and there is an } \leq_{\mathfrak{K}_\mu}\text{-increasing continuous sequence } \langle M_i : i \leq \kappa \rangle \text{ and } p \in \mathcal{S}(M_\kappa) \text{ such that } M_{2i+2} \text{ is universal over } M_{2i+1} \text{ and } p \upharpoonright M_{2i+2} \text{ does } \mu\text{-split over } M_{2i+1}\}$
- (b)  $\kappa_{\text{sp}}(\mathfrak{K}_\mu) := \{\kappa : \kappa \text{ regular} \leq \mu \text{ and there is an } \leq_{\mathfrak{K}_\mu}\text{-increasing continuous sequence } \langle M_i : i \leq \kappa \rangle \text{ and } p \in \mathcal{S}(M_\kappa) \text{ which } \mu\text{-splits over } M_i \text{ for each } i < \kappa \text{ and } M_{2i+2} \text{ is universal over } M_{2i+1}\}$ .

*Proof of 7.9.* By playing with  $\text{EM}(I, \Phi)$ , (or see Claim [Sh 394, 5.7=5.7tex] and Definition [Sh 394, 4.9=4.4tex]).  $\square_{7.9}$

**7.11 Question:** Can we omit assumption 7.8(c) (see below so  $\chi = \text{LS}(\mathfrak{K})$ )?

**7.12 Theorem.** For some cardinal  $\lambda_* < \chi$  and a cardinal  $\lambda_{**} < \beth_{1,1}(\lambda_*^{+\omega})$  above  $\lambda_*$ ,  $\mathfrak{K}$  is categorical in every cardinal  $\lambda \geq \lambda_{**}$  but in no  $\lambda \in (\lambda_*, \lambda_{**})$  provided that:

- $\circledast_{\mathfrak{K}}^{\mu, \chi}$  (a)  $K$  is an a.e.c. cateogorical in  $\mu$
- (b)  $\mathfrak{K}$  has amalgamation and JEP in every  $\lambda < \aleph_\chi, \lambda \geq \text{LS}(\mathfrak{K})$
- (c)  $\chi$  is a limit cardinal,  $\text{cf}(\chi) > \text{LS}(\mathfrak{K})$ , and for arbitrarily large  $\lambda < \chi$  the sequence  $\langle 2^{\lambda^{+n}} : n < \omega \rangle$  is increasing
- (d)  $\mu > \beth_{1,1}(\lambda)$  for every  $\lambda < \chi$  hence  $\mu \geq \aleph_\chi$
- (e) every  $M \in K_{< \aleph_\chi}$  has a  $\leq_{\mathfrak{K}}$ -extension in  $K_\mu$ .

*Remark.* 1) Concerning [Sh 394] note

- (a) there the central case was  $\mathfrak{K}$  with full amalgamation (not just below  $\chi \ll \mu!$ ), trying to concentrate on the difficulty of lack of localness,
- (b) when we use clause (e) this is just to get the “ $M \in K_\mu$  is  $\lambda$ -saturated” , this is where we use 7.8
- (c) we demand “ $\text{cf}(\chi) > \text{LS}(\mathfrak{K})$ ” to prove locality.

2) We rely on Chapter III and Chapter IV in the end.

3) The assumption (e) of 7.12 follows if  $\mathfrak{K}$  has amalgamation in every  $\lambda' \leq \beth_{1,1}(\lambda)$  for  $\lambda < \chi$  which is a reasonable assumption.

4) Most of the proof works even if we weaken the assumption (a) to “ $\mathfrak{K}$  is strongly solvable in  $\mu$ ” and even weakly solvable, i.e. up to  $\square_7$ , we continue in and see more [Sh:F782].

5) Theorem 7.12 also continue Kolman-Shelah [KlSh 362], [Sh 472], as its assumptions are proved there.

*Proof.* Let  $\kappa = \text{LS}(\mathfrak{K})$  and let  $\Phi \in \Upsilon_\kappa^{\text{or}}[\mathfrak{K}]$  be as guaranteed by 0.9(1) hence

- (\*)<sub>1</sub> if  $I \in K_\chi^{\text{lin}}$  then  $\text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$  belongs to  $K_\lambda$  for  $\lambda \geq \text{LS}(\mathfrak{K})$  (and in the strongly solvable case,  $I \in K_\mu^{\text{lin}} \Rightarrow \text{EM}_{\tau(\mathfrak{K})}(I, \Phi) \in K_\mu^x$ )

and

- (\*)<sub>2</sub> if  $I \subseteq J$  are from  $K^{\text{lin}}$  then  $\text{EM}_{\tau(\mathfrak{K})}(I, \Phi) \leq_{\mathfrak{K}} \text{EM}_{\tau(\mathfrak{K})}(J, \Phi)$ .

Also



- (\*)<sub>3</sub>  $\langle \mathcal{S}_{\mathfrak{K}}(M) : M \in \mathfrak{K}_{<\aleph_\chi} \rangle$  has the reasonable basic properties.

[Why? See III. 600-0.12 and III. 600-0.12A because  $\mathfrak{K}_{<\aleph_\chi}$  has the amalgamation property by clause (b) of the assumption  $\otimes_{\mathfrak{K}}^{\mu, \chi}$ .]

- (\*)<sub>4</sub> if  $M \in K_\mu$  then  $M$  is  $\chi$ -saturated (hence  $\chi$ -model homogeneous).

[Why? We shall prove that: if  $\text{LS}(\mathfrak{K}) \leq \lambda < \chi$  and  $M \in K_\mu^x$  then  $M$  is  $\lambda^+$ -saturated. We shall show that all the assumptions of 7.8 with  $(\mu, \chi, \lambda)$  there standing for  $(\mu, \aleph_\chi, \lambda)$  here hold. Let us check; clause (a) of 7.8 means “ $\mathfrak{K}$  is categorical in  $\mu$ ” (or is strongly solvable) which holds by clause (a) of  $\otimes_{\mathfrak{K}}^{\mu, \chi}$ . Clause (b) of 7.8 says that  $\text{LS}(\mathfrak{K}) \leq \lambda < \aleph_\chi \leq \mu$  and  $2^{2^\lambda} \leq \mu$ ; the first holds because of the way  $\lambda$  was chosen above and the second holds as clause (d) of  $\otimes_{\mathfrak{K}}^{\mu, \chi}$  says that  $\mu > \beth_{1,1}(\lambda)$  and  $\mu \geq \aleph_\chi$ . Clause (c)( $\alpha$ ) of 7.8 holds as  $\lambda^{+\lambda^{+4}} < \aleph_{\lambda^{+5}}$  which is  $< \aleph_\chi$  as  $\chi$  is a limit

cardinal and  $\aleph_\chi$  here plays the role of  $\chi$  there. Clause (d) of 7.8 says  $\aleph_{\geq \partial} \neq \emptyset$  for every cardinal  $\partial$ , holds by  $(*)_1$  above. Lastly, clause (e) of 7.8 holds more exactly clauses  $(e)(\beta)(i) + (iii)$  hold by clauses (b) + (e) of  $\otimes_{\aleph}^{\mu, \chi}$  and they suffice.

We have shown that all the assumptions of 7.8 holds, hence its conclusion, which says, as  $M \in K_\mu$ , that  $M$  is  $\lambda^+$ -saturated. The “ $\chi$ -model homogeneous” holds by III. 600-0.19.

$(*)_5$  if  $M \leq_{\aleph} N$  are from  $K_\mu^x$  then  $M \prec_{\mathbb{L}_{\infty, \chi}[\aleph]} N$ .

[Why? Obvious by  $(*)_4$ .]

$(*)_6$  if  $\lambda \in (\kappa, \chi)$  and  $I \in K_{\geq \lambda}^{\text{lin}}$  is  $\lambda$ -wide then  $\text{EM}_{\tau(\aleph)}(I, \Phi)$  is  $\lambda$ -saturated; moreover, if  $I^+ \in K_\chi^{\text{lin}}$  is wide over  $I$  then every  $p \in \mathcal{S}(\text{EM}_{\tau(K)}(I, \Phi))$  is realized in  $\text{EM}_{\tau(\aleph)}(I^+, \Phi)$ .

[Why? By 1.14, its assumption “ $\Phi$  satisfies the conclusion of 1.12” holds by  $(*)_5$ , (or as in [Sh 394, 6.8=6.5tex]). The “moreover” is immediate by  $(*)_4$  as in the proof of  $\otimes(d)$  inside the proof of 7.2 above or see the proof of  $(*)_{10}$  below.]



$(*)_7$   $\aleph$  is stable in  $\lambda$  when  $\kappa \leq \lambda < \chi$ .

[Why? Recalling clause (e) of the assumption of 7.12, by Claim 7.8 or more accurately  $(*)_0$  in its proof as we have proved (in the proof of  $(*)_4$ ) that the assumptions of 7.8 holds with  $(\mu, \chi, \lambda)$  there standing for  $(\mu, \aleph_\chi, \lambda)$  here.]

$(*)_8$  if  $\lambda \in [\kappa, \chi)$  and  $M \in K_\lambda^x$  then there is  $N \in \aleph_\lambda$  which is  $(\lambda, \aleph_0)$ -brimmed over  $M$

[Why? By  $(*)_7$  and III. 600-0.22 (1)(b) remembering the amalgamation, clause (b) of the assumption of the theorem.]

$(*)_9$  if  $\langle M_\alpha : \alpha \leq \lambda \rangle$  is  $\leq_{\aleph}$ -increasing continuous,  $\kappa \leq \|M_\lambda\| \leq \lambda < \chi$ , then no  $p \in \mathcal{S}_{\aleph}(M_\lambda)$  satisfies  $p \restriction M_{i+1}$  does  $\lambda$ -split over  $M_i$  for every  $i < \lambda$ .

[Why? Otherwise we get contradiction to stability in  $\lambda$ , i.e.  $(*)_7$ , see in IV. 705-gr.6 (1B), using amalgamation (using the tree  $\theta^{>2}$  when  $\theta = \min\{\partial : 2^\partial > \lambda\}$ ; also we can prove it as in the proof of case 2 inside the proof of 7.8.)

We could use more

$(*)_{10}$  if  $I_1, I_2$  are wide linear orders of cardinality  $\lambda \in (\kappa, \chi)$  and  $I_2$  is wide over  $I_1$  so  $I_1 \subseteq I_2$  and  $M_\ell = \text{EM}_{\tau(\aleph)}(I_\ell, \Phi)$ , then  $M_2$  is universal over  $M_1$  and even brimmed over  $I_1$ , even  $(\lambda, \partial)$ -brimmed for any regular  $\partial < \lambda$ .

[Why? As  $I_2$  is wide over  $I_1$ , we can find a sequence  $\langle J_\gamma : \gamma < \lambda \rangle$  of pairwise disjoint subsets of  $I_2 \setminus I_1$  such that each  $J_\gamma$  is a convex subset of  $I_2$  and in  $J_\gamma$  there is a monotonic sequence  $\langle t_{\gamma,n} : n < \omega \rangle$  of members. Let  $\langle \gamma_\varepsilon : \varepsilon < \lambda \times \partial \rangle$  list  $\lambda$ , and let  $I_{2,0} = I_1$  and  $I_{2,1+\varepsilon} = I_2 \setminus \cup \{J_{\gamma_\zeta} : \zeta \in [1+\varepsilon, \lambda \times \partial)\}$  and  $M'_\zeta = \text{EM}_{\tau(\mathfrak{K})}(I_{2,\varepsilon}, \Phi)$ . So  $\langle M'_\zeta : \zeta \leq \lambda \times \partial \rangle$  is  $\leq_{\mathfrak{K}}$ -increasing continuous sequence of members of  $K_\lambda$ ; first member  $M_1$ , last member  $M_2$ .

By III. 600-0.22 (4)(b) it is enough to prove that if  $\varepsilon < \lambda \times \partial$  and  $p \in \mathcal{S}(M_\varepsilon)$  then  $p$  is realized in  $M_{\varepsilon+1}$ . As  $I_1$  is wide of cardinality  $\lambda$  so is  $I_{2,\varepsilon}$  hence  $M'_\varepsilon$  is saturated. Also for each  $\varepsilon$  we can find a linear order  $I_{2,\varepsilon}^+$  of cardinality  $\lambda$  such that  $I_{2,\varepsilon+1} \subseteq I_{2,\varepsilon}^+$  and  $J_\varepsilon^+ = I_{2,\varepsilon+1}^+ \setminus I_{2,\varepsilon}$  is a convex subset of  $I_{2,\varepsilon+1}^+$  and is a wide linear order of cardinality  $\lambda$  which is strongly  $\aleph_0$ -homogeneous, (recall  $J_{\gamma_\varepsilon} \subseteq J_{\gamma_\varepsilon}^+$  is infinite). So in  $M_{\varepsilon+1}^+ = \text{EM}_{\tau(\mathfrak{K})}(I_{2,\varepsilon+1}^+, \Phi)$  every  $p \in \mathcal{S}(M_\varepsilon)$  is realized (as  $I_{2,\varepsilon+1}^+$  is wide over  $I_{2,\varepsilon}$  as  $J_\varepsilon^+$  is wide of cardinality  $\lambda$ ), moreover realized in  $M'_{\varepsilon+1}$  (why? by the strong  $\aleph_0$ -homogeneous every element and even finite sequence from  $M_{\varepsilon+1}^+$  can be mapped by some automorphism of  $M_{\varepsilon+1}^+$  over  $M_\varepsilon$  into  $M_{\varepsilon+1}$ ). As said above, this suffices.]



⊗<sub>1</sub>  $\chi_*$  is well defined  $\in (\kappa, \chi)$  where

$$\begin{aligned} \chi_* = \text{Min}\{ & \theta : \kappa < \theta < \chi \text{ and for every saturated} \\ & M \in \mathfrak{K}, \text{ if } \theta \leq \|M\| < \chi, \text{ every} \\ & p \in \mathcal{S}(M) \text{ is } \theta\text{-local, see Definition 7.4(2)} \}. \end{aligned}$$

[Why? By 7.2 which we apply with  $(\mu, \kappa)$  there standing for  $(\chi, \kappa)$  here recalling  $\kappa = \text{LS}(\mathfrak{K})$ ; this is O.K. as: clause (a) in 7.2 holds by clause (c) of the assumption here, clause (b) in 7.2 holds by clause (b) of the assumption here as  $\chi \leq \aleph_\chi$ . Lastly, clause (c) in 7.2 easily follows by  $(*)_6$  above.]





⊗<sub>2</sub> if  $\lambda \in (\kappa, \chi)$  and  $\langle M_i : i \leq \delta \rangle$  is  $\leq_{\mathfrak{K}_\lambda}$ -increasing continuous,  $M_{i+1}$  is  $\leq_{\mathfrak{K}}$ -universal over  $M_i$  for  $i < \delta$  then  $M_\delta$  is saturated and moreover every  $p \in \mathcal{S}(M_\delta)$  does not  $\lambda$ -split over  $M_\alpha$  for some  $\alpha < \delta$ .

[Why? For  $i \leq \delta$  let  $I_i$  be the linear order  $\lambda \times \lambda \times (1+i)$  and  $M'_i = \text{EM}_{\tau(\mathfrak{K})}(I_i, \Phi)$ . So  $\langle M'_i : i \leq \delta \rangle$  is  $\leq_{\mathfrak{K}_\lambda}$ -increasing continuous. Also for  $i \leq \delta, \zeta \leq \lambda$  let  $I_{i,\zeta} = \lambda \times \lambda \times (1+i) + \lambda \times \zeta$  and  $M'_{i,\zeta} = \text{EM}_{\tau(\mathfrak{K})}(I_{i,\zeta}, \Phi)$ , so for each  $i < \delta$  the sequence  $\langle M'_{i,\zeta} : \zeta \leq \lambda \rangle$  is  $\leq_{\mathfrak{K}_\lambda}$ -increasing continuous,  $M'_{i,0} = M'_i, M'_{i,\lambda} = M'_{i+1}$ . Now for  $i < \delta, \zeta < \lambda$  every  $p \in \mathcal{S}(M_{i,\zeta})$  is realized in  $M'_{i,\zeta+1}$  by  $(*)_6$  and the definition of type, varying the linear order. By III. 600-0.22 (4)(b) the model  $M'_{i+1}$  is  $\leq_{\mathfrak{K}_\lambda}$ -universal over  $M'_i$  and by Definition III. 600-0.21 the

models  $M'_\delta$  and  $M_\delta$  are  $(\lambda, \text{cf}(\delta))$ -brimmed hence by III. 600-0.22 (3) are isomorphic. But  $M'_\delta$  is saturated by  $(*)_6$ , hence  $M_\delta$  is saturated.

What about the “moreover”? (Note that if  $\lambda = \lambda^{\text{cf}(\delta)}$  then  $(*)_9$  does not cover it.) We can find easily  $\langle I''_\alpha : \alpha \leq \lambda \times \delta + 1 \rangle$  such that:

- (a)  $I''_\alpha$  is a linear order of cardinality  $\lambda$  into which  $\lambda$  can be embedded
- (b)  $I''_\alpha$  is increasing continuous with  $\alpha$
- (c)  $I''_\alpha$  is an initial segment of  $I''_\beta$  for  $\alpha < \beta \leq \delta + 1$
- (d)  $I''_{\alpha+1}$  has a subset of order types  $\lambda \times \lambda$  whose convex hull is disjoint to  $I''_\alpha$
- (e) if  $\alpha \leq \beta < \lambda \times \delta$  and  $s \in I''_{\lambda \times \delta + 1} \setminus I''_{\lambda \times \delta}$  then there is an automorphism  $\pi_{\alpha, \beta, s}$  of  $I''_{\lambda \times \delta + 1}$  mapping  $I''_{\beta+1}$  onto  $I''_{\lambda \times \delta}$  and is over  $I''_\alpha \cup \{t \in I''_{\lambda \times \delta + 1} : s \leq I''_{\lambda \times \delta + 1} t\}$ .

Let  $M''_\alpha = \text{EM}_{\tau(\mathfrak{K})}(I''_\alpha, \Phi)$ , so  $\langle M''_{\lambda \times \alpha} : \alpha \leq \delta \rangle$  has the properties of  $\langle M'_\alpha : \alpha \leq \delta \rangle$ , i.e. every  $p \in \mathcal{S}(M''_\alpha)$  is realized in  $M''_{\alpha+1}$  hence  $M''_{\alpha+\lambda}$  is  $\leq_{\mathfrak{K}_\lambda}$ -universal over  $M''_\alpha$ . So (easily or see III. 600-0.22, III. 600-0.21) there is an isomorphism  $f$  from  $M_\delta$  onto  $M''_{\lambda \times \delta}$  such that  $M''_{\lambda \alpha} \leq_{\mathfrak{K}} f(M_{\alpha+1}) \leq M''_{\lambda \alpha + 2}$ . So it suffices to prove the “moreover” for  $\langle M''_{\lambda \times \alpha} : \alpha \leq \delta \rangle$ , equivalently for  $\langle M''_\alpha : \alpha \leq \lambda \times \delta \rangle$ . Let  $p \in \mathcal{S}(M''_{\lambda \times \delta})$  so some  $a \in M''_{\lambda \times \delta + 1}$  realizes it, hence for some  $t_0 < \dots < t_{n-1}$  from  $I''_{\lambda \times \delta + 1}$  and  $\tau_\Phi$ -term  $\sigma(x_0, \dots, x_{n-1})$  we have  $a = \sigma^{\text{EM}(I''_{\lambda \times \delta + 1}, \Phi)}(a_{t_0}, \dots, a_{t_{n-1}})$ , it follows that for some  $m \leq n$  we have  $t_\ell \in I''_{\lambda \times \delta} \Leftrightarrow \ell < m$  and let  $\alpha < \lambda \times \delta$  be such that  $\{t_\ell : \ell < m\} \subseteq I''_\alpha$ ; if  $m = n$  choose any  $t_n \in I''_{\lambda \times \delta + 1} \setminus I''_{\lambda \times \delta}$ . If  $\beta \in (\alpha, \lambda \times \delta)$  and  $\text{tp}_{\mathfrak{K}}(a, M''_\delta, M''_{\delta+1})$  does  $\lambda$ -split over  $M''_\beta$  then  $\pi' := \pi_{\beta, \beta, t_m}$  is an automorphism of  $I''_{\lambda \times \delta + 1}$  mapping  $I''_{\beta+1}$  onto  $I''_{\lambda \times \delta}$  and is over  $I''_\beta \cup \{s \in I''_{\lambda \times \delta + 1} : t_m \leq I''_{\lambda \times \delta + 1} s\}$  hence it is the identity on  $\{t_\ell : \ell < n\}$ ; now  $\pi'$  induces an automorphism  $\hat{\pi}'$  of  $\text{EM}_{\tau(\mathfrak{K})}(I''_{\lambda \times \delta + 1}, \Phi)$ , so clearly it maps  $a$  to itself and maps  $\text{tp}_{\mathfrak{K}}(a, M''_{\beta+1}, M''_{\lambda \times \delta + 1})$  to  $\text{tp}_{\mathfrak{K}}(a, M''_{\lambda \times \delta}, M''_{\lambda \times \delta + 1})$  and it maps  $M''_\beta$  onto itself, hence also  $\text{tp}_{\mathfrak{K}}(a, M''_{\beta+1}, M''_{\delta+1})$  does  $\lambda$ -split over  $M''_\beta$ . So if for some  $\beta \in (\alpha, \lambda \times \delta)$ , the type  $\text{tp}_{\mathfrak{K}}(a, M''_\delta, M''_{\delta+1})$  does not  $\lambda$ -split over  $M''_\beta$  we get the desired conclusion, but otherwise this contradicts  $(*)_9$ .]

⊗<sub>3</sub> If  $\lambda \in [\chi_*, \chi)$  and  $M \in K_\lambda$  is saturated and  $p \in \mathcal{S}(M)$  then for some  $N$  we have:

- (a)  $N \leq_{\mathfrak{K}} M$
- (b)  $N \in K_{\chi_*}$  is saturated
- (c)  $p$  does not  $\chi_*$ -split over  $N$
- (d)  $p$  does not  $\lambda$ -split over  $N$  (follows by (a),(b),(c)).

[Why  $\otimes_3$  holds? For clauses (a),(b),(c) use  $\otimes_2$  or just  $(*)_9$ ; for clause (d) use localness, i.e. recall  $\otimes_1$  and Definition 7.4.]

- $\otimes_4$  Assume  $\lambda \in [\kappa, \chi)$  and  $M_1 \leq_{\mathfrak{K}} M_2 \leq_{\mathfrak{K}} M_3$  are members of  $K$ ,  $M_2$  is  $\lambda^+$ -saturated and  $p \in \mathcal{S}(M_3)$ . If  $N_\ell \leq_{\mathfrak{K}} M_\ell$  is from  $K_{\leq \lambda}$  and  $p \restriction M_{\ell+1}$  does not  $\lambda$ -split over  $N_\ell$  for  $\ell = 1, 2$  then  $p$  does not  $\lambda$ -split over  $N_1$ .

[Why? Easy manipulations. Without loss of generality  $N_1 \leq_{\mathfrak{K}} N_2$  as we can increase  $N_2$ . So for some pair  $(M_4, a)$  we have  $M_3 \leq_{\mathfrak{K}} M_4$ ,  $a \in M_4$  and  $p = \mathbf{tp}_{\mathfrak{K}}(a, M_3, M_4)$ . Assume  $\alpha < \lambda^+$  and let  $\bar{b}, \bar{c} \in {}^\alpha(M_3)$  be such that  $\mathbf{tp}_{\mathfrak{K}}(\bar{b}, N_1, M_3) = \mathbf{tp}_{\mathfrak{K}}(\bar{c}, N_1, M_3)$ . As  $M_2$  is  $\lambda^+$ -saturated and  $N_2 \leq_{\mathfrak{K}} M_2 \leq_{\mathfrak{K}} M_3$  we can find  $\bar{b}', \bar{c}' \in {}^\alpha(M_2)$  such that  $\mathbf{tp}_{\mathfrak{K}}(\bar{b}' \wedge \bar{c}', N_2, M_3) = \mathbf{tp}_{\mathfrak{K}}(\bar{b} \wedge \bar{c}, N_2, M_3)$  using III. 600-0.19. Hence  $\mathbf{tp}_{\mathfrak{K}}(\bar{b}', N_1, M_3) = \mathbf{tp}_{\mathfrak{K}}(\bar{b}, N_1, M_3) = \mathbf{tp}_{\mathfrak{K}}(\bar{c}, N_1, M_3) = \mathbf{tp}_{\mathfrak{K}}(\bar{c}', N_1, M_3)$ .

By the choice of  $(M_4, a)$  and the assumption on  $N_1$  that  $p \restriction M_2$  does not  $\lambda$ -split over  $N_1$  we get

$$\mathbf{tp}_{\mathfrak{K}}(\langle a \rangle^{\bar{b}'}, N_1, M_4) = \mathbf{tp}_{\mathfrak{K}}(\langle a \rangle^{\bar{c}'}, N_1, M_4).$$

Clearly  $\mathbf{tp}_{\mathfrak{K}}(\bar{b}', N_2, M_3) = \mathbf{tp}_{\mathfrak{K}}(\bar{b}, N_2, M_3)$  hence by the choice of  $(M_4, a)$  and the assumption on  $N_2$  that  $p$  does not  $\lambda$ -split over  $N_2$  we have  $\mathbf{tp}_{\mathfrak{K}}(\langle a \rangle^{\bar{b}'}, N_2, M_4) = \mathbf{tp}_{\mathfrak{K}}(\langle a \rangle^{\bar{b}}, N_2, M_4)$  hence by monotonicity

$$\mathbf{tp}_{\mathfrak{K}}(\langle a \rangle^{\bar{b}'}, N_1, M_4) = \mathbf{tp}_{\mathfrak{K}}(\langle a \rangle^{\bar{b}}, N_1, M_4).$$

Similarly

$$\mathbf{tp}_{\mathfrak{K}}(\langle a \rangle^{\bar{c}'}, N_1, M_4) = \mathbf{tp}_{\mathfrak{K}}(\langle a \rangle^{\bar{c}}, N_1, M_4).$$

As equality of types is transitive  $\mathbf{tp}_{\mathfrak{K}}(\langle a \rangle^{\bar{c}}, N_1, M_4) = \mathbf{tp}_{\mathfrak{K}}(\langle a \rangle^{\bar{c}'}, N_1, M_4) = \mathbf{tp}_{\mathfrak{K}}(\langle a \rangle^{\bar{b}'}, N_1, M_4) = \mathbf{tp}_{\mathfrak{K}}(\langle a \rangle^{\bar{b}}, N_1, M_4)$  as required.]

- $\otimes_5$  Assume  $I_3 = I_0 + I'_1 + I'_2$  are wide linear orders of cardinality  $\lambda$  where  $\chi > \lambda > \kappa$  and let  $I_\ell = I_0 + I'_\ell$  for  $\ell = 1, 2$  and  $M_\ell = \text{EM}_{\tau(\mathfrak{K})}(I_\ell, \Phi)$  for  $\ell = 0, 1, 2, 3$ . If  $\ell \in \{1, 2\}$  and  $\bar{a} \in {}^{\lambda >}(M_\ell)$  then  $\mathbf{tp}_{\mathfrak{K}_\lambda}(\bar{a}, M_{3-\ell}, M_3)$  does not  $\lambda$ -split over  $M_0$ , (moreover if  $\mathbf{tp}_{\mathfrak{K}_\lambda}(\bar{a}, M_0, M_3)$  does not  $\lambda$ -split over  $N \in K_{\leq \lambda}$  then also  $\mathbf{tp}_{\mathfrak{K}_\lambda}(\bar{a}, M_{3-\ell}, M_3)$  does not  $\lambda$ -split over  $N$ ).

[Why? For  $\ell = 2$ , if the desired conclusion fails we get a contradiction as in the proof of  $\otimes_2$  so for  $\ell = 2$  we get the conclusion. For  $\ell = 1$  if the desired conclusion fails (but it holds for  $\ell = 2$ ) we get a contradiction to categoricity in  $\mu$  by the order property (by 1.5).]

- $\otimes_6$  If  $\lambda \in (\chi_*, \chi)$ ,  $\delta < \lambda^+$ ,  $\langle M_i : i \leq \delta \rangle$  is  $\leq_{\mathfrak{K}_\lambda}$ -increasing continuous and  $i < \delta \Rightarrow M_i$  saturated then  $M_\delta$  is saturated.

[Why? Let  $N \leq_{\mathfrak{K}} M_\delta$ ,  $\|N\| < \lambda$  and  $p \in \mathcal{S}(N)$ . If  $\text{cf}(\delta) > \|N\|$  this is easy so assume  $\text{cf}(\delta) \leq \|N\|$  hence  $\text{cf}(\delta) < \lambda$  and without loss of generality  $\delta = \text{cf}(\delta)$  and choose a cardinal  $\theta$  such that  $\text{LS}(\mathfrak{K}) < \chi_* + |\text{cf}(\delta)| + \|N\| \leq \theta < \lambda$  and  $\|N\|^+ < \lambda \Rightarrow \|N\| < \theta$  and let  $q \in \mathcal{S}(M_\delta)$  extend  $p$ , exist as  $\mathfrak{K}_{\leq \lambda}$  has amalgamation.

Now for every  $X \subseteq M_\delta$  of cardinality  $\leq \theta$  we can choose  $N_i \leq_{\mathfrak{K}} M_i$  by induction on  $i \leq \delta$  such that  $N_i \in K_\theta$  is saturated, is  $\leq_{\mathfrak{K}}$ -increasing continuous with  $i$  and  $N_i$  is  $\leq_{\mathfrak{K}}$ -universal over  $N_j$  and includes  $(X \cup N) \cap M_i$  when  $i = j + 1$ . So by  $\otimes_2$  (we justify the choice of  $N_i$  for limit  $i$  and) the model  $N_\delta$  is saturated, so if  $\|N\|^+ < \lambda$  then  $N \leq_{\mathfrak{K}} N_\delta$ ,  $N_\delta$  is saturated of cardinality  $\theta > \|N\|$  so we are done as  $N_\delta \leq_{\mathfrak{K}} M_\delta$ , so without loss of generality  $\lambda = \|N\|^+$  hence  $\lambda = \theta^+$ .

Also for some  $\alpha_* < \delta$  and  $N_* \leq_{\mathfrak{K}} M_{\alpha_*}$  of cardinality  $\theta$ , the type  $q$  does not  $\theta$ -split over  $N_*$ . [Why? Otherwise we choose  $(N_i, N_i^+)$  by induction on  $i \leq \delta$  such that  $N_i \leq_{\mathfrak{K}} N_i^+$  are from  $K_\theta$ ,  $N_i \leq_{\mathfrak{K}} M_i$ ,  $N_i^+ \leq_{\mathfrak{K}} M_\delta$ ,  $N_i$  is  $\leq_{\mathfrak{K}}$ -increasing continuous,  $N_i$  is  $\leq_{\mathfrak{K}}$ -universal over  $N_j$  if  $i = j + 1$  and  $q \upharpoonright N_i^+$  does  $\theta$ -split over  $N_i$  and  $\cup\{N_j^+ \cap M_i : j < i\} \subseteq N_i$ . In the end we get a contradiction to  $\otimes_2$ .]

We can find  $N' \leq_{\mathfrak{K}} M_{\alpha_*}$  from  $K_{\chi_*}$  such that  $q \upharpoonright M_{\alpha_*}$  does not  $\theta$ -split over  $N'$ , (why? by  $\otimes_3$ ) and without loss of generality  $N' \leq_{\mathfrak{K}} N_*$  and  $N' \leq_{\mathfrak{K}} N$ . Also  $q$  does not  $\theta$ -split over  $N'$  (why? by applying  $\otimes_4$ , with  $\theta, N_*, M_{\alpha_*}, M_\delta$  here standing for  $\lambda, M_1, M_2, M_3, N_1, N_2$  there; or use  $N' = N_*$ ).

By  $(*)_6$  as  $M_{\alpha_*}$  is saturated without loss of generality  $M_{\alpha_*} = \text{EM}_{\tau(\mathfrak{K})}(\lambda, \Phi)$  and for  $\varepsilon < \lambda$  let  $M_{\alpha_*, \varepsilon} = \text{EM}_{\tau(\mathfrak{K})}(\theta \times \theta \times (1 + \varepsilon), \Phi)$ , so  $M_{\alpha_*, \varepsilon} \in K_\theta$  is saturated and is brimmed over  $M_{\alpha_*, \zeta}$  when  $\varepsilon = \zeta + 1$  by  $(*)_{10}$ . So for each  $\varepsilon < \lambda$  there is  $a_\varepsilon \in M_{\alpha_*, \varepsilon+1}$  realizing  $q \upharpoonright M_{\alpha_*, \varepsilon}$ . Also without loss of generality  $M_\delta \leq_{\mathfrak{K}} \text{EM}_{\tau(\mathfrak{K})}(\lambda + \lambda, \Phi)$  as in the proof of  $\otimes_2$  or by  $(*)_{10}$ , now for some  $\varepsilon(*) < \lambda$  we have  $N \leq_{\mathfrak{K}} \text{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$  and  $N_* \leq_{\mathfrak{K}} \text{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$  where  $I_0 = \theta \times \theta \times (1 + \varepsilon(*))$  and  $I_2 = [\lambda, \lambda + \varepsilon(*)) \cup I_0$ . Let  $I_1 = \theta \times \theta \times \zeta(*)$  where  $\zeta(*) \in (\varepsilon(*), \lambda)$  is large enough such that  $a_{\varepsilon(*)} \in \text{EM}_{\tau(\mathfrak{K})}(I_1, \Phi)$ , e.g.  $\zeta(*) = 1 + \varepsilon(*) + 1$  and let  $I_3 = I_1 \cup I_2 \subseteq \lambda + \lambda$ . Let  $M'_\ell = \text{EM}_{\tau(\mathfrak{K})}(I_\ell, \Phi)$  for  $\ell = 0, 1, 2, 3$ .

Now we apply  $\otimes_5$ , the “moreover” with  $\theta, I_0, I_1, I_2, I_1 \setminus I_0, I_2 \setminus I_0, a_{\varepsilon(*)}, N'$  here standing for  $\lambda, I_0, I_1, I_2, I'_1, I'_2, \bar{a}, N$  there and we conclude that  $\text{tp}_{\mathfrak{K}_\lambda}(a_{\varepsilon(*)}, M'_2, M'_3)$  does not  $\theta$ -split over  $N'$ .

As  $N' \leq_{\mathfrak{K}} M'_0 \leq_{\mathfrak{K}} M'_2$  also the type  $q' := \text{tp}_{\mathfrak{K}_\lambda}(a_{\varepsilon(*)}, M'_2, M'_3)$  does not  $\theta$ -split over  $N'$ . Let us sum up:  $q \upharpoonright M'_2, q'$  belong to  $\mathcal{S}_{\mathfrak{K}_\lambda}(M'_2)$ , does not  $\theta$ -split over  $N'$ ,  $N' \in K_{\chi_*}$  and  $\chi_* \leq \theta$ . Also  $N' \leq_{\mathfrak{K}_*} M'_0 \leq_{\mathfrak{K}_*} M'_2$ , the model  $M'_0$  is  $\theta$ -saturated and  $q \upharpoonright M_{\alpha_*} = q' \upharpoonright M_{\alpha_*}$ . By the last two sentences obviously  $q = q'$  (it may be more transparent to consider  $q \upharpoonright (\leq \chi_*) = q' \upharpoonright (\leq \chi_*)$ ), so we are done proving  $\otimes_6$ .]

$\otimes_7$  If  $\lambda \in (\chi_*, \chi)$  then the saturated  $M \in \mathfrak{K}_\lambda$  is superlimit.

[Why? By  $\otimes_6$ , (existence by  $(*)_6$ , the non-maximality by  $(*)_6$ + uniqueness; you may look at [Sh 394, 6.7=6.4tex](1).]



Now we have arrived to the main point

- ⊙<sub>1</sub> If  $\lambda \in (\chi_*, \chi)$  then  $\mathfrak{s}_\lambda$  is a full good  $\lambda$ -frame,  $K_{\mathfrak{s}_\lambda}$  categorical where  $\mathfrak{s}_\lambda$  is defined by
- (a)  $\mathfrak{K}_{\mathfrak{s}_\lambda} = \mathfrak{K}_\lambda \upharpoonright \{M \in K_\lambda : M \text{ saturated}\}$
  - (b)  $\mathcal{S}_{\mathfrak{s}_\lambda}^{\text{bs}}(M) = \mathcal{S}_{\mathfrak{s}_\lambda}^{\text{na}}(M) := \{\mathbf{tp}_{\mathfrak{s}}(a, M, N) : M \leq_{\mathfrak{K}_\lambda} N \text{ and } a \in N \setminus M\}$  for  $M \in K_{\mathfrak{s}_\lambda}$
  - (c)  $p \in \mathcal{S}_{\mathfrak{s}_\lambda}^{\text{bs}}(M_2)$  does not fork over  $M_1$  when  $M_1 \leq_{\mathfrak{s}_\lambda} M_2$  and for some  $M \leq_{\mathfrak{K}} M_1$  of cardinality  $\chi_*$ , the type  $p$  does not  $\chi_*$ -split over  $N$ .

[Why? We check the clauses of Definition III. 600-1.1 .

$K_{\mathfrak{s}_\lambda}$  is categorical:

By III. 600-0.34  (1) and  $\otimes_7$ .

Clause (A), Clause (B): By  $\otimes_7$  recalling that there is a saturated  $M \in K_{\mathfrak{s}_\lambda}$  (and it is not  $<_{\mathfrak{s}_\lambda}$ -maximal) by  $(*)_6$  and trivially recalling III. 600-0.34 , of course.

Clause (C): By categoricity and  $(*)_6$  clearly no  $M \in K_{\mathfrak{s}_\lambda}$  is maximal; amalgamation and JEP holds by clause (b) of the assumption of the claim.

Clause (D)(a),(b): By the definition.

Clause (D)(c): Density is obvious; in fact  $\mathfrak{s}_\lambda$  is full.

Clause (D)(d): (bs - stability).

Easily  $\mathcal{S}_{\mathfrak{s}_\lambda}(M) = \mathcal{S}_{\mathfrak{K}_\lambda}(M)$  which has cardinality  $\leq \lambda$  by the moreover in  $(*)_6$ .

Clause (E)(a): By the definition.

Clause (E)(b): Monotonicity (of non-forking).

By the definition of “does not  $\chi_*$ -split”.

Clause (E)(c): Local character.

Why? Let  $\langle M_\alpha : \alpha \leq \delta \rangle$  be  $\leq_{\mathfrak{s}_\lambda}$ -increasing continuous,  $\delta < \lambda^+$  and  $q \in \mathcal{S}_{\mathfrak{s}_\lambda}^{\text{bs}}(M_\delta)$ . Using the third paragraph of the proof of  $\otimes_6$  for  $\theta = \chi_*$ , for some  $\alpha_* < \delta$  and  $N_* \leq_{\mathfrak{s}_\lambda} M_{\alpha_*}$  of cardinality  $\theta$  the type  $q$  does not  $\theta$ -split over  $N_*$ . So clearly  $q$  does not fork over  $M_{\alpha_*}$  (for  $\mathfrak{s}_\lambda$ ), as required.

Clause (E)(d): Transitivity of non-forking.

By  $\otimes_4$ .

Clause (E)(e): Uniqueness.

Holds by the choice of  $\chi_*$ , i.e. by  $\otimes_1$ .



Clause (E)(f): Symmetry.

Why? Let  $M_\ell$  for  $\ell \leq 3$  and  $a_0, a_1, a_2$  be as in (E)(f)' in III. 600-1.16E. We can find a  $\leq_{\mathfrak{K}}$ -increasing continuous sequence  $\langle M_{0,\alpha} : \alpha \leq \lambda^+ \rangle$  such that  $M_{0,0} = M_0$ ,  $M_{0,\alpha+1}$  is  $\leq_{\mathfrak{s}_\lambda}$ -universal over  $M_{0,\alpha}$  and without loss of generality  $M_{0,\alpha} = \text{EM}_{\tau(\mathfrak{K})}(\gamma_\alpha, \Phi)$  so is  $\leq_{\mathfrak{K}}$ -increasing continuous, and  $\lambda$  divides  $\gamma_\alpha$ .

By (E)(g) proved below we can find  $a_\alpha^\ell \in M_{0,\alpha+1}$  realizing  $\mathbf{tp}_{\mathfrak{s}_\lambda}(a_\ell, M_0, M_{\ell+1})$  such that  $\mathbf{tp}_{\mathfrak{s}_\lambda}(a_\alpha^\ell, M_{0,\alpha}, M_{0,\alpha+1})$  does not fork over  $M_0 = M_{0,0}$ , for  $\ell = 1, 2$ . We can find  $N_* \leq_{\mathfrak{K}} M_0$  of cardinality  $\chi_*$  such that  $\mathbf{tp}_{\mathfrak{s}_\lambda}(\langle a_1, a_2 \rangle, M_0, M_3)$  does not  $\chi_*$ -split over  $N_*$  so  $N_* \leq_{\mathfrak{K}} M_{0,0}$ .

Then as in 1.5 we get a contradiction (recalling III. 600-1.16E).

Clause (E)(g): Extension existence.

If  $M \leq_{\mathfrak{s}_\lambda} N$  and  $p \in \mathcal{S}_{\mathfrak{s}_\lambda}^{\text{bs}}(M) = \mathcal{S}_{\mathfrak{K}}^{\text{na}}(M)$ , then  $p$  does not  $\chi_*$ -split over  $M_*$  for some  $M_* \leq_{\mathfrak{K}} M$  of cardinality  $\chi_*$  by  $\otimes_3$ . Let  $M^* \in K_{\chi^*}$  be such that  $M_* \leq_{\mathfrak{K}} M^* \leq_{\mathfrak{K}} M$  and  $M^*$  is  $\leq_{\mathfrak{K}}$ -universal over  $M_*$ . As  $M, N \in K_{\mathfrak{s}_\lambda} \subseteq K_\lambda$  are saturated there is an isomorphism  $\pi$  from  $M$  onto  $N$  over  $M^*$  and let  $q = \pi(p)^+$ .

Now  $q \upharpoonright M = p$  by  $\otimes_1$  as both are from  $\mathcal{S}_{\mathfrak{K}}^{\text{na}}(M)$ , does not  $\chi_*$ -split over  $M_*$  and has the same restriction to  $M^*$ .

Clause (E)(h): Follows by III. 600-1.16A (3),(4) recalling  $\mathfrak{s}_\lambda$  is full.

Clause (E)(i): Follows by III. 600-1.15.

So we have finished proving “ $\mathfrak{s}_\lambda$  is a good  $\lambda$ -frame.”]

$\odot_2$  If  $\lambda \in (\chi_*, \chi)$  then  $\mathfrak{K}^{\mathfrak{s}_\lambda}$  is  $\mathfrak{K} \upharpoonright \{M : M \text{ is } \lambda\text{-saturated}\}$ .

[Why? Should be clear.]

$\odot_3$   $\lambda_*$  is well defined where

$$\lambda_* = \text{Min}\{\lambda : \chi_* < \lambda < \chi \text{ and } 2^{\lambda^{+n}} < 2^{\lambda^{+n+1}} \text{ for every } n < \omega\}.$$

[Why? By clause (c) of the assumption.]

Let  $\Theta = \{\lambda_*^{+n} : n < \omega\}$ .

$\odot_4$   $\mathfrak{s}_\lambda$  is weakly succsesful for  $\lambda \in \Theta$ .

[Why? Recalling that “ $\mathfrak{s}_\lambda$  categorical”, by Definition IV. 705-stg.0A, Definition III. 600-nu.1 and Observation III. 600-nu.13.1 (b) this means that if  $(M, N, a) \in K_{\mathfrak{s}_\lambda}^{3, \text{bs}}$  then for some  $(M_1, N_1, a) \in K_{\mathfrak{s}_\lambda}^{3, \text{uq}}$  we have  $(M, N, a) \leq_{\mathfrak{s}_\lambda}^{\text{bs}} (M_1, N_1, a)$  (see Definition III. 600-nu.1A). Toward contradiction, assume that this fails. Let  $\langle M_\alpha : \alpha < \lambda^+ \rangle$  be  $\leq_{\mathfrak{s}_\lambda}$ -increasing continuous,  $M_{\alpha+1}$  is brimmed over  $M_\alpha$  for  $\alpha < \lambda^+$  such that  $M_0 = M$ . Now directly by the definitions (as in III§5, see more in Chapter VII) we can find  $\langle M_\eta, f_\eta : \eta \in {}^{\lambda^+}2 \rangle$  such that:

- (a) if  $\eta \triangleleft \nu \in {}^{\lambda^+}2$  then  $M_\eta \leq_{\mathfrak{s}_\lambda} M_\nu$
- (b) if  $\eta \in {}^{\lambda^+}2$  then  $f_\eta$  is a one-to-one function from  $M_{\ell g(\eta)}$  to  $M_\eta$  over  $M_0 = M$  such that  $\rho \triangleleft \eta \Rightarrow f_\rho \subseteq f_\eta$  and  $f_\eta(M_{\ell g(\eta)}) \leq_{\mathfrak{s}_\lambda} M_\eta$  in fact  $f_0 = \text{id}_M$  and  $(M, N, a) \leq_{\mathfrak{s}_\lambda}^{\text{bs}} (f_\eta(M_{\ell g(\eta)}), M_\eta, a) \in K_{\mathfrak{s}}^{\text{bs}}$
- (c) if  $\nu = \eta^\wedge \langle \ell \rangle \in {}^{\lambda^+}2$  then  $M_\nu$  is brimmed over  $M_\eta$
- (d) if  $\eta \in {}^{\lambda^+}2$  then  $f_{\eta^\wedge \langle 0 \rangle}(M_{\ell g(\eta)+1}) = f_{\eta^\wedge \langle 1 \rangle}(M_{\ell g(\eta)+1})$
- (e) if  $\eta \in {}^{\lambda^+}2$  then there is no triple  $(N, f_0, f_1)$  such that  $f_{\eta^\wedge \langle 1 \rangle}(M_{\ell g(\eta)+1}) \leq_{\mathfrak{s}} N$ , and  $f_\ell$  is a  $\leq_{\mathfrak{s}_\lambda}$ -embedding of  $M_{\eta^\wedge \langle \ell \rangle}$  into  $N$  over  $f_{\eta^\wedge \langle \ell \rangle}(M_{\ell g(\eta)+1})$  for  $\ell = 0, 1$  and  $f_0 \upharpoonright M_\eta = f_1 \upharpoonright M_\eta$ .

Having carried the induction by renaming without loss of generality  $\eta \in {}^{\lambda^+}2 \Rightarrow f_\eta = \text{id}_{M_{\ell g(\eta)}}$ . Now  $M_* := \cup \{M_\alpha : \alpha < \lambda^+\}$ ; it belongs to  $\mathfrak{s}_{\lambda^+}$  and is saturated and for  $\eta \in {}^{\lambda^+}2$  let  $M_\eta := \cup \{M_{\eta \upharpoonright \alpha} : \alpha < \lambda^+\}$  so  $M_* \leq_{\mathfrak{s}_{\lambda^+}} M_\eta \in K_{\mathfrak{s}_{\lambda^+}}$ . But  $\chi$  is a limit cardinal so also  $\lambda^+ \in (\kappa, \chi)$  so let  $N_* \in K_{\mathfrak{s}_{\lambda^+}}$  be  $\leq_{\mathfrak{s}_{\lambda^+}}$ -universal over  $M_*$ , so for every  $\eta \in {}^{\lambda^+}2$  there is an  $\leq_{\mathfrak{s}_\lambda}$ -embedding  $h_\eta$  of  $M_\eta$  into  $N_*$  over  $M_*$ . But  $2^\lambda < 2^{\lambda^+}$  by the choice of  $\lambda_*$  so by I. 88r-0.wD we get a contradiction to clause (e).]

⊙<sub>5</sub> for  $\lambda \in \Theta$ , if  $M \in K_{\lambda^+}^{\mathfrak{s}_\lambda}$  is saturated above  $\lambda$  for  $K^{\mathfrak{s}_\lambda}$ , then  $M$  is saturated for  $\mathfrak{K}$ .

[Why? Should be clear and implicitly was proved above.]

⊠<sub>1</sub>  $\text{NF}_{\mathfrak{s}_\lambda}$  is well defined and is a non-forking relation on  $\mathfrak{K}_{\mathfrak{s}_\lambda}$  respecting  $\mathfrak{s}_\lambda$  (for  $\lambda \in \Theta$ ).

[Why? By III§6 as  $\mathfrak{s}_\lambda$  is a weakly successful good  $\lambda$  frame.]

⊠<sub>2</sub>  $\mathfrak{s}_\lambda$  is a good<sup>+</sup>  $\lambda$ -frame (for  $\lambda \in \Theta$ ).

[Recalling Definition IV. 705-stg.1, assume that this fails so there are  $\langle M_i, N_i : i < \lambda^+ \rangle$  and  $\langle a_{i+1} : i < \lambda^+ \rangle$ , as there, i.e.  $a_{i+1} \in M_{i+2} \setminus M_{i+1}$ ,  $\text{tp}_{\mathfrak{s}_\lambda}(a_{i+1}, M_{i+1}, M_{i+2})$  does not fork over  $M_0$  for  $\mathfrak{s}_\lambda$ , but  $\text{tp}_{\mathfrak{s}_\lambda}(a_{i+1}, N_0, M_{i+1})$  forks over  $M_0$ . Also, recalling Definition IV. 705-stg.1 the model  $M = \cup \{M_i : i < \lambda^+\}$  is saturated for  $\mathfrak{K}_{\lambda^+}^{\mathfrak{s}_\lambda}$  hence by ⊙<sub>5</sub> for  $\mathfrak{K}$ , so it belongs to  $K_{\mathfrak{s}_{\lambda^+}}$ .

We can find an isomorphism  $f_0$  from  $M$  onto  $\text{EM}_{\tau(\mathfrak{K})}(\lambda^+, \Phi)$ , by  $(*)_6$ . By the “moreover” from  $(*)_6$ , more exactly by  $(*)_{10}$  we can find a  $\leq_{\mathfrak{K}}$ -embedding  $f_1$  of  $N = \cup \{N_i : i < \lambda^+\}$  into  $\text{EM}_{\tau(\mathfrak{K})}(\lambda \times \lambda, \Phi)$  extending  $f_0$ . As we can increase the  $N_i$ ’s without loss of generality  $f_1$  is onto  $\text{EM}_{\tau(\mathfrak{K})}(\lambda \times \lambda, \Phi)$ . We can find  $\delta < \lambda^+$  such that  $N_\delta = \text{EM}_{\tau(\mathfrak{K})}(u, \Phi)$  where  $u = \{\lambda\alpha + \beta : \alpha, \beta < \delta\}$ . By  $a_{\delta+1}$  we get a contradiction to ⊗<sub>5</sub>.]

□<sub>3</sub> Let  $\lambda \in \Theta$

- (α)  $\leq_{\mathfrak{s}_\lambda}^*$  is a partial order on  $K_{\lambda^+}^{\text{nice}}[\mathfrak{s}_\lambda] = K_{\mathfrak{s}_{\lambda^+}}$  and  $(K_{\mathfrak{s}_{\lambda^+}}, \leq_{\mathfrak{s}_\lambda}^*)$  satisfies the demands on a.e.c. except possibly smoothness, see III§7
- (β) if  $M \in K_{\lambda^+}$  is saturated and  $p \in \mathcal{S}_{\mathfrak{K}}(M)$  then for some pair  $(N, a)$  we have  $M \leq_{\mathfrak{s}_\lambda}^* N$  and  $a \in N$  realizes  $p$
- (γ) if  $M \in K_{\lambda^+}$  is saturated then some  $N$  satisfies:
  - (a)  $N \in K_{\lambda^+}$  is saturated
  - (b)  $N$  is  $\leq_{\mathfrak{K}}$ -universal over  $M$
  - (c)  $M \leq_{\mathfrak{s}_\lambda}^* N$
- (δ)  $\mathfrak{s}_\lambda$  is successful.

[Why? Clause (α):

We know that both  $K_{\lambda^+}^{\text{nice}}[\mathfrak{s}_\lambda]$  and  $K_{\mathfrak{s}_{\lambda^+}}$  are the class of saturated  $M \in K_\lambda$ . The rest holds by III§7, §8.

Clause (β):

By  $\otimes_3$  we can find  $M_* \leq_{\mathfrak{K}} M$  of cardinality  $\chi_*$  such that  $p$  does not  $\chi_*$ -split over it (equivalently does not  $\lambda^+$ -split over it).

Let  $\langle M_\alpha : \alpha < \lambda^+ \rangle$  be  $\leq_{\mathfrak{s}_\lambda}$ -increasing continuous such that  $M_{\alpha+1}$  is brimmed over  $M_\alpha$  for  $\mathfrak{s}_\lambda$  for every  $\alpha < \lambda^+$  and  $M_* \leq_{\mathfrak{K}} M_0$  (so  $\|M_*\| < \|M_0\|$  otherwise we would require  $M_0$  is brimmed over  $M_*$ ). Hence  $\cup\{M_\alpha : \alpha < \lambda^+\} \in K_{\lambda^+}$  is saturated (by  $\odot_5$ ) so without loss of generality is equal to  $M$ . We can choose  $a_*, N_\alpha (\alpha < \lambda)$  such that  $\langle N_\alpha : \alpha < \lambda^+ \rangle$  is  $\leq_{\mathfrak{s}_\lambda}$ -increasing continuous,  $M_\alpha \leq_{\mathfrak{s}_\lambda} M_\alpha$ ,  $\text{NF}_{\mathfrak{s}_\lambda}(M_\alpha, N_\alpha, M_\beta, M_\beta)$  for  $\alpha < \beta < \lambda^+$ ,  $N_{\alpha+1}$  is brimmed over  $M_{\alpha+1} \cup N_\alpha$  and  $\text{tp}_{\mathfrak{s}_\lambda}(a, N_0, M_0) = p \upharpoonright M_0$  so  $a \in N_0$ . Let  $N = \cup\{N_\alpha : \alpha < \lambda^+\}$  so again  $N \in K_{\lambda^+}$  is saturated (equivalently  $N \in K_{\lambda^+}^{\text{nice}}[\mathfrak{s}_\lambda]$ ) and  $M \leq_{\mathfrak{K}} N$  and even  $M \leq_{\mathfrak{s}_\lambda}^* N$  (by the definition of  $\leq_{\mathfrak{s}_\lambda}^*$ ). For each  $\alpha < \lambda^+$  we have  $\text{NF}_{\mathfrak{s}_\lambda}(M_0, N_0, M_\alpha, N_\alpha)$  but  $\text{NF}_{\mathfrak{s}_\lambda}$  respect  $\mathfrak{s}_\lambda$  hence  $\text{tp}_{\mathfrak{s}_\lambda}(a, M_\alpha, N_\alpha)$  does not fork over  $M_0$  hence by the definition of  $\mathfrak{s}_\lambda$  the type  $\text{tp}_{\mathfrak{s}_\lambda}(a, M_\alpha, N_\alpha)$  does not  $\lambda$ -split over  $M_*$  hence  $\text{tp}_{\mathfrak{s}_\lambda}(a, M_\alpha, N_\alpha) = p \upharpoonright M_\alpha$ . As this holds for every  $\alpha < \lambda^+$ , by the choice of  $\chi_*$ , i.e. by  $\otimes_1$  clearly  $a$  realizes  $p$ .

Clause (γ):

By clause (β) as in the proofs in III§4; that is, we choose  $N \in K_{\lambda^+}$  which is  $\leq_{\mathfrak{K}}$ -universal over  $M$ . We now try to choose  $(M_\alpha, f_\alpha, N_\alpha)$  by induction on  $\alpha < \lambda^+$  such that:  $M_0 = M, N_0 = N, f_0 = \text{id}_M, M_\alpha$  is  $\leq_{\mathfrak{s}_\lambda}^*$ -increasing continuous,  $N_\alpha$  is  $\leq_{\mathfrak{K}}$ -increasing continuous,  $f_\alpha$  is a  $\leq_{\mathfrak{K}}$ -embedding of  $M_\alpha$  into  $N_\alpha$ ,  $f_\alpha$  is  $\subseteq$ -increasing continuous with  $\alpha$  and  $\alpha = \beta + 1 \Rightarrow f_\alpha(M_\alpha) \cap N_\beta \neq f_\beta(M_\beta)$ .

For  $\alpha = 0$ ,  $\alpha$  limit no problems. If  $\alpha = \beta + 1$  and  $f_\alpha(M_\alpha) = N_\alpha$  we are done and otherwise use clause  $(\beta)$ . But by Fodor lemma we cannot carry the induction for every  $\alpha < \lambda^+$ , so we are done proving  $(\gamma)$ .

Clause  $(\delta)$ :

We should verify the conditions in Definition IV. 705-stg.0A. Now clause (a) there, being weakly successful, holds by  $\odot_4$ . As for clause (b) there, it suffices to prove that if  $M_1, M_2 \in K_{\lambda^+}^{\text{nice}}[\mathfrak{s}_\lambda] = K_{\mathfrak{s}_\lambda^+}$  and  $M_1 \leq_{\mathfrak{K}} M_2$  then  $M_1 \leq_{\mathfrak{s}_\lambda}^* M_2$  which means: if  $\langle M_\alpha^\ell : \alpha < \lambda^+ \rangle$  is  $\leq_{\mathfrak{s}_\lambda}$ -increasing continuous,  $M_{\alpha+1}^\ell$  is brimmed over  $M_\alpha^\ell$  with  $M_\ell = \cup \{M_\alpha^\ell : \alpha < \lambda^+\}$ , then for some club  $E$  of  $\lambda^+$  for every  $\alpha < \beta$  from  $E$ ,  $\text{NF}_{\mathfrak{s}_\lambda}(M_\alpha^1, M_\alpha^2, M_\beta^1, M_\delta^2)$ .

By clause  $(\gamma)$  there is  $N \in K_{\mathfrak{s}_\lambda^+}$  such that  $M_1 \leq_{\mathfrak{s}_\lambda^+}^* N$  (hence  $M_1 \leq_{\mathfrak{K}} N$ ) and  $N$  is  $\leq_{\mathfrak{K}^{\mathfrak{s}_\lambda}}$ -universal over  $M_1$ . So without loss of generality  $M_2 \leq_{\mathfrak{K}} N$  but by III. 600-ne.3 (3) all this implies  $M_1 \leq_{\lambda^+}^* M_2$ . So we are done proving  $\Box_3$ .

$\Box_4$   $\mathfrak{s}_{\lambda^+}$  is the successor of  $\mathfrak{s}_\lambda$  for  $\lambda \in \Theta$ .

[Why? Now by  $\Box_3$  the good frame  $\mathfrak{s}_\lambda$  is successful; by IV. 705-stg.3 we know that  $\mathfrak{s}_\lambda^+$  is a well defined good  $\lambda^+$ -frame. Clearly  $K_{\mathfrak{s}_\lambda(+)}$  is the class of saturated  $M \in \mathfrak{K}_{\lambda^+}$ , by  $\odot_5$ , see the definitions in III. 600-ne.1, III. 600-rg.7 (5). But  $\mathfrak{s}_\lambda$  is good $^+$  by  $\Box_2$  so by IV. 705-stg.3B we know that  $\leq_{\mathfrak{s}_\lambda(+)} = \leq_{\lambda^+}^* [\mathfrak{s}_\lambda]$  is equal to  $\leq_{\mathfrak{K}} \upharpoonright K_{\mathfrak{s}_\lambda(+)}$ , so  $\mathfrak{K}_{\mathfrak{s}_\lambda(+)} = \mathfrak{K}_{\mathfrak{s}_\lambda^+}$ . As both  $\mathfrak{s}_\lambda(+)$  and  $\mathfrak{s}_{\lambda^+}$  are full, clearly  $\mathcal{S}_{\mathfrak{s}_\lambda(+)}^{\text{bs}} = \mathcal{S}_{\mathfrak{s}_{\lambda^+}}^{\text{bs}}$ . For  $M_1 \leq_{\mathfrak{s}_\lambda(+)} M_2 \leq_{\mathfrak{s}_\lambda(+)} M_3$  and  $a \in M_3 \setminus M_2$ , comparing the two definitions of “ $\text{tp}_{\mathfrak{K}_{\mathfrak{s}_\lambda(+)}(a, M_2, M_1)}$  does not fork over  $M_1$ ” they are the same. So we are done.]

$\Box_5$   $\mathfrak{s}_{\lambda_*^{+\omega}}$  is the limit of  $\langle \mathfrak{s}_{\lambda_*^+}^n : n < \omega \rangle$ .

[Why? Should be clear.]

$\Box_6$   $\mathfrak{s}_\lambda$  satisfies the hypothesis IV. 705-12.1 of IV§12 if  $\lambda \in \Theta \setminus \lambda_*^{+3}$  holds.

[Why? By  $\Box_2, \Box_3, \Box_4$  and IV. 705-12D.1.]

Hence

$\Box_7$   $\mathfrak{s}_{\lambda_*}$  is beautiful  $\lambda_*^{+\omega}$ -frame.

[Why? By IV. 705-12b.14 and IV. 705-12f.16A.]

$\Box_8$   $K[\mathfrak{s}_{\lambda_*^{+\omega}}]$  is categorical in one  $\chi > \lambda_*^{+\omega}$  iff it is categorical in every  $\chi > \lambda^{+\omega}$ .

[Why? By IV. 705-12f.16A (d),(e).]

$\square_9$  if  $\lambda \geq \beth_{1,1}(\lambda_*^{+\omega})$  then  $\mathfrak{K}_\lambda = \mathfrak{K}_\lambda[\mathfrak{s}_{\lambda_*^{+\omega}}]$ .

[Why? The conclusion  $\supseteq$  is obvious. For the other inclusion let  $M \in K_\lambda$ , now by the definition of class in the left, it is enough to prove that  $M$  is  $(\lambda_*^{+\omega})^+$ -saturated. But otherwise by the omitting type theorem for a.e.c., i.e. by 0.9(1),(d), (or see [Sh 394, 8.6=X1.3A]) there is such a model  $M' \in K_\mu$ , contradiction to  $(*)_4$ .]

By  $\square_8 + \square_9$  we are done.

$\square_{7.12}$

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